ON THE COMPUTABILITY OF GRAPHONS

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ABSTRACT. We investigate the relative computability of exchangeable binary relational data when presented in terms of the distribution of an invariant measure on graphs, or as a graphon in either L^1 or the cut distance. We establish basic computable equivalences, and show that L^1 representations contain fundamentally more computable information than the other representations, but that $0'$ suffices to move between computable such representations. We show that $0'$ is necessary in general, but that in the case of random-free graphons, no oracle is necessary. We also provide an example of an L^1-computable random-free graphon that is not weakly isomorphic to any graphon with an a.e. continuous version.

1. Introduction

A sequence of random variables is exchangeable when its distribution does not depend on the ordering of its elements. A well-known theorem of de Finetti shows that infinite exchangeable sequences of random variables are conditionally independent and identically distributed (i.i.d.), meaning that, by introducing a new random variable and conditioning on its value, any apparent dependence between the random variables in the exchangeable sequence is removed and all the variables have the same distribution.

Exchangeable sequences are models for homogeneous data sets and serve as building blocks for statistical models with more interesting dependency structures. Conditional independence and exchangeability are also central to the probabilistic programming. Infinite exchangeable sequences arise naturally in functional probabilistic programming languages. Indeed, any sequence of evaluations of a closure is a finite prefix of an infinite exchangeable sequence. The sequence is even manifestly conditionally i.i.d.: conditioning on the closure itself, every evaluation is independent and identically distributed.

The more interesting phenomenon is the existence of (potentially stateful) probabilistic code with the property that repeated evaluations produce an exchangeable sequence yet no existing variable renders the sequence conditionally i.i.d. Exchangeability, nevertheless, licenses a programmer or compiler to commute and even prune these repeated evaluations. These types of transformations are central to several probabilistic programming systems, including Church [GMRBT08] and Venture [MSP14].

As described in [AAFRR17], a fundamental question for probabilistic programming is whether or not support for exchangeability is in some sense necessary on the grounds of efficiency or even computability. By de Finetti's theorem, an infinite exchangeable sequence of random variables $X_1, X_2, \ldots$ admits a representation $X_j = f(\theta, \xi_j)$ a.s., for all $i \in \mathbb{N}$, where $f : \Omega \times [0, 1] \to [0, 1]$ is a measurable function and $\theta, \xi_1, \xi_2, \ldots$ are random variables, with $\xi_1, \xi_2, \ldots$ i.i.d. and independent also from θ. Clearly, there are many such f. On the other hand, any such f could be taken to be a concrete "representation" of the conditional independence underlying the sequence $X_1, X_2, \ldots$, and so it is natural to ask when some such f is computable. (Indeed, it suffices to establish the computability of the distribution of random probability measure $\nu = \mathbf{P}[f(\theta, \xi_1) \in \cdot | \theta]$.) This question was studied in the setting of exchangeable sequences of real random variables in [FR12], which established the computability of the distribution of ν, and showed that it was even uniformly computable from the distribution of the exchangeable sequence, yielding an effectivization of de Finetti's theorem that acts like a program transformation.

Here we study a generalization of de Finetti's theorem to two dimensions, and in particular, the binary symmetric case of the Aldous–Hoover theorem for two-dimensional jointly exchangeable arrays. Our focus will be on the computability of various representations of the distribution of these two-dimensional arrays, with a special emphasis on the representation of graphons. (The function $f(\theta, \cdot)$ is a one-dimensional analogue of a graphon. For more details on graphons and exchangeable arrays, see [Lov12] and [Kal05], respectively.) The situation here is more complicated, and depends on the choice of metric. Several of the standard metrics are computably equivalent to each other, but we find that one natural way of expressing the relevant measurable object — corresponding to the so-called edit distance — is not computable from the distribution of the exchangeable data itself, unlike in the one-dimensional case. This mapping is possible using the halting problem $0'$ as an oracle, and we provide an example showing that this dependence is necessary. On the other hand, for a natural subclass, the so-called random-free case, one can recover the graphon in the edit distance metric from the distribution of the array.

In the case of computable distributions on binary symmetric exchangeable arrays, all these results fall short of establishing that some graphon is computable on a measure one set of points. In fact, one cannot hope for such representations. We show that, in the case of computable distributions on arrays, there need not be any graphon generating the array that is almost-everywhere continuous, providing a fundamental topological impediment to computable representations that would render it continuous on a measure-one set.

It follows that the two-dimensional setting is fundamentally different from the one-dimensional one. There need not be a representation of an exchangeable array that exposes the conditional independence inherent in the exchangeable array, even if the exchangeable array itself has some computable representation. In some special cases, we can compute an L^1 representation, which exposes the conditional independence, but only allows us to represent the distribution of the exchangeable array up to an arbitrarily small, but nonzero, error term. Our results suggest that probabilistic programming languages may need to have special support for probabilistic symmetries, such as exchangeability in the case of arrays.

1.1. **Summary of main results.** In this paper, we examine four main representations of invariant measures on graphs, as elements of complete pseudo-metric spaces, as we describe in Section 2. We consider the space of invariant measures with a metric equivalent to weak convergence, and the space of graphons under the metrics d_1, $d_\square$, and $\delta_\square$. The distance d_1 is the L^1 pseudo-metric on graphons, and is closely related to edit distance on graphs. The $\delta_\square$ distance between two graphons can be seen as measuring the difference

in distribution between graphs sampled from them, and is obtained from the simpler $d_\square$ upon taking the infimum under measure-preserving transformations of the graphon.

We establish the key computable relationships between these representations in Section 3, using the framework of computable analysis. In particular, we show that there is a computable equivalence between names for invariant measures and $\delta_\square$-names for the corresponding graphons, and show that the identity map from $d_\square$-names to $\delta_\square$-names has a computable section.

While every d_1-name is also a $d_\square$-name, in Section 4 we show how to transform a computable $d_\square$-name into a d_1-name using the halting problem $0'$. In Section 5 we show that $0'$ is necessary, by constructing a graphon with a computable $d_\square$-name such that $0'$ can be computed from any d_1-name — although d_1 names are computable from $d_\square$ names in the random-free case. Finally, we establish in Section 6 that there is a d_1-computable graphon that is not even a.e. continuous in any topology consisting of Borel sets.

1.2. **Notation.** Let λ denote Lebesgue measure on $\mathbb{R}$ (though for notational convenience, we will often use λ to also refer to Lebesgue measure on $\mathbb{R}^2$, etc.). For $n \in \mathbb{N}$, write $[n] := \{0, \ldots, n-1\}$. All logarithms, written log, will be in base 2.

For $e \in \mathbb{N}$, let $\{e\}$ denote the partial computable function $\mathbb{N} \to \mathbb{N}$ given by computer program e. For $n \in \mathbb{N}$ write $\{e\}(n)\downarrow$ to denote that this program halts on input n, and $\{e\}(n)\uparrow$ to denote that it does not halt. For $s \in \mathbb{N}$ write $\{e\}_s(n)\downarrow$ to denote that this program has halted after at most s steps, and $\{e\}(n)_s\uparrow$ to denote that it has not yet halted after s steps.

Write $0' := \{e \in \mathbb{N} : \{e\}(0)\downarrow\}$ to denote the halting set. Recall that a set $X \subseteq \mathbb{N}$ is (co-)c.e. complete when it (its complement) is computably enumerable and any computably enumerable set admits a computable many-one reduction, called an *m-reduction*, to it (its complement, respectively). For example, $0'$ is c.e. complete.

2. Graphons and invariant measures on graphs

In this section we provide the basic definitions and results about the two main objects of interest in this paper, graphons and invariant measures on graphs.

2.1. **Graphons.** We will formulate most of our results in terms of graphons, both for concreteness and simplicity of notation. In this subsection we summarize the standard notions that we will need. For more details on graphons, their basic properties, and notation, we refer the reader to [Lov12, Chapter 7], from which most of the definitions in this subsection are borrowed. In many cases,

analogous notions and results were developed earlier in terms of exchangeable arrays; for details on this connection and the history, see [Aus08] and [DJ08].

Definition 2.1. A **graphon** is a symmetric measurable function $W \colon [0,1]^2 \to [0,1]$. Let $\mathcal{W}_0$ denote the set of all graphons.

Three classes of graphons play a special role in this paper.

Definition 2.2. A graphon G is **random-free** if it is $\{0,1\}$-valued a.e., in other words, when $\lambda\big(G^{-1}(\{0,1\})\big) = 1$.

Definition 2.3. Let $\mathcal{T}$ be a topology on $[0,1]$. A graphon G is **almost everywhere (a.e.) continuous** with respect to $\mathcal{T}$ if there is a set $X \subseteq [0,1]^2$ such that $\lambda(X) = 1$ and $G|_X$ is continuous with respect to $\mathcal{T}|_X$.

Definition 2.4. A graphon G is a **step function** if there is a (finite) measurable partition P of $[0,1]$ such that for every pair of parts $p, q \in P$, the graphon G is constant on $p \times q$. Write $\mathcal{S}$ to denote the class of step functions whose underlying partition divides $[0,1]$ into some finite number of equally-sized intervals, and whose range is contained in the rationals. Write $\mathcal{S}_0$ to denote the subset of its random-free graphons. Each of $\mathcal{S}$ and $\mathcal{S}_0$ admits a straightforward computable enumeration.

One can associate to each finite graph (on $[n]$ for some $n \in \mathbb{N}$) a random-free step function graphon, as we now describe, such that the space of graphons is the completion of the finite graphs (embedded this way) under an appropriate pseudometric on finite graphs.

Definition 2.5. Let G be a graph with vertex set $[n]$, for some $n \in \mathbb{N}$. Define the step function graphon $W_G \colon [0,1]^2 \to \{0,1\}$ to be such that $W_G(x,y) = 1$ if and only if there is an edge between $\lfloor nx \rfloor$ and $\lfloor ny \rfloor$ in G. In other words, W_G is equal to 1 on $[i/n, (i+1)/n) \times [j/n, (j+1)/n)$ if there is an edge from i to j, and 0 otherwise. Observe that $W_G \in \mathcal{S}_0$.

We will use three key pseudometrics on the space of graphons. We begin by describing the cut norm $\|\cdot\|_\square$, which will allow us to define $d_\square$ and then $\delta_\square$, the latter of which is closely related to subsampling. The third pseudometric is d_1, more closely related to edit distance and the L^1 norm.

Definition 2.6. The **cut norm** of a symmetric measurable function $F \colon [0,1]^2 \to [-1,1]$ is defined by

$$\left\| F \right\|_\square := \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} F(x,y) \, dx \, dy \right| \qquad (1)$$

where S and T range over measurable sets. For graphons U and W, define $d_\square(U, W) := \|U - W\|_\square$.

It is straightforward to verify that $\|\cdot\|_\square$ is a norm on $\mathcal{W}_0$, and that $d_\square$ is a pseudometric on $\mathcal{W}_0$.

As we will see, this cut norm is too coarse for many of our purposes.

Definition 2.7. The L^1 **norm** of a symmetric measurable function $F\colon [0,1]^2 \to [-1,1]$ is defined by

$$\|G\|_1 := \textstyle\int |F(x,y)|\ dx\,dy. \tag{2}$$

For graphons U and W, define $d_1(U,W) := \|U - W\|_1$.

It is a standard fact that the L^1 norm is a norm on $\mathcal{W}_0$, and that d_1 is a pseudometric on $\mathcal{W}_0$.

While the cut norm is much coarser than the L^1 norm, they do agree on the notion of norm zero. The following easy lemma follows from the fact that if $\|W\|_\square = 0$ then $W = 0$ a.e.

Lemma 2.8. *If W is a graphon, then $\|W\|_1 = 0$ if and only if $\|W\|_\square = 0$.*

This lemma implies that the pseudometrics d_1 and $d_\square$ can be thought of as metrics on the same quotient space, namely $\mathcal{W}_0/\{(G,H)\ :\ d_1(G,H) = 0\}$, even though the metrics they induce on this space are very different.

As we will see in §2.2, there is a standard way to associate to each graphon an invariant measure on countable graphs and given two graphons we would like to have a condition equivalent to the corresponding invariant measures being the same. However, it is easy to see, by applying a non-trivial measure-preserving transformation to any non-constant random-free graphon, that there are graphons which give rise to the same distribution but which are very far in either d_1 or $d_\square$. Hence we will need an even coarser notion of distance, which we now define.

Definition 2.9. Let W be a graphon and let $\varphi\colon ([0,1],\lambda) \to ([0,1],\lambda)$ be a measure-preserving map. Define W^φ to be the graphon satisfying

$$W^\varphi(x,y) = W\big(\varphi(x),\varphi(y)\big) \tag{3}$$

for all $x,y \in [0,1]$.

Definition 2.10. For graphons U and W, define

$$\delta_\square(U,W) := \inf_\varphi d_\square(U,W^\varphi), \tag{4}$$

where the infimum is taken over all measure-preserving maps of $([0,1],\lambda)$ to itself.

The following standard result will be important when we consider the computability of the representations of graphons in these various metric spaces.

Lemma 2.11. *The set $\mathcal{S}$ is dense in $(\mathcal{W}_0, d_1)$, and its subset $\mathcal{S}_0$ is dense in $(\mathcal{W}_0, d_\square)$ and $(\mathcal{W}_0, \delta_\square)$.*

Proof. The density of step functions in d_1 is a standard measure-theoretic fact. The density of $\mathcal{S}_0$ in $\delta_\square$ follows from [Lov12, Theorem 11.52]. This implies that

$$\left\{ W^\varphi \ : \ W \in \mathcal{S}_0 \text{ and } \varphi \text{ is a measure-preserving map} \right\} \tag{5}$$

is dense in $d_\square$. But for every $W \in \mathcal{S}_0$, measure-preserving map φ, and $\epsilon > 0$, there is an element $V \in \mathcal{S}_0$ such that

$$d_\square(W^\varphi, V) < \epsilon. \tag{6}$$

Hence $\mathcal{S}_0$ is also dense in $d_\square$. $\qquad\square$

We will later need the following definition.

Definition 2.12. A graphon W is **twin-free** if for each pair of distinct points $x, y \in [0, 1]$, the functions $z \mapsto W(x, z)$ and $z \mapsto W(y, z)$ disagree on a set of positive Lebesgue measure.

2.2. **Invariant measures on graphs.** Invariant measures on graphs with underlying set $\mathbb{N}$ are the main object of study in this paper. In the probability theory literature, one often studies exchangeable arrays rather than their distributions, but here we focus on their distribution as we will be interested in the measures rather than the random variables, and so that we can avoid certain technicalities and notational difficulties.

Definition 2.13. Let $\mathcal{G} \subseteq \{0, 1\}^{\mathbb{N}^2}$ denote the space of adjacency matrices of symmetric irreflexive graphs with underlying set $\mathbb{N}$. A probability measure μ on the space $\mathcal{G}$ is called an **invariant measure on graphs** if $\mu(A) = \mu(\sigma^{-1}(A))$ for all Borel $A \subseteq \mathcal{G}$ and all permutations $\sigma \colon \mathbb{N} \to \mathbb{N}$.

We will use the term *invariant measure* to refer to invariant measures on graphs.

An important subclass of the invariant measures are those that are extreme.

Definition 2.14. An invariant measure μ is **extreme** if there do not exist invariant measures ν and π such that $\mu = \alpha\nu + (1 - \alpha)\pi$ for some $\alpha \in (0, 1)$.

In our setting, the extreme measures coincide with the ergodic ones (with respect to permutations of $\mathbb{N}$).

Graphons naturally give rise to extreme invariant measures on graphs, via the distribution of the countably infinite random graph obtained by sampling from the graphon, as we now describe.

Definition 2.15. Let W be a graphon and let S be a countable set. Let $\langle \zeta_i \rangle_{i \in S}$ be an i.i.d. collection of uniform $[0, 1]$-valued random variables. Consider the random graph $\widehat{\mathbb{G}}(S, W)$ with vertex set S where for all distinct $i, j \in S$, there is

an edge between i and j independently, with probability $W(\zeta_i, \zeta_j)$. For $n \in \mathbb{N}$, we write $\widehat{\mathbb{G}}(n, W)$ to refer to $\widehat{\mathbb{G}}([n], W)$. When H is a finite graph, we write $\widehat{\mathbb{G}}(S, H)$ to refer to $\widehat{\mathbb{G}}(S, W_H)$. Finally, let $\mathbb{G}(S, W)$ denote the distribution of $\widehat{\mathbb{G}}(S, W)$.

For $0 < p < 1$, if W is the constant graphon $W \equiv p$, then $\mathbb{G}(\mathbb{N}, W)$ is the distribution of an Erdős–Rényi random graph.

The following lemma is standard.

Proposition 2.16 ([Lov12, Theorem 11.52])**.** *If W is a graphon, then $\mathbb{G}(\mathbb{N}, W)$ is an extreme invariant measure on graphs.*

Conversely, every extreme invariant measure arises from a graphon.

Proposition 2.17 ([Lov12, Theorem 11.52])**.** *If μ is an extreme invariant measure on graphs then there is some graphon W such that $\mathbb{G}(\mathbb{N}, W)$ and μ are the same distribution.*

It is then natural to ask when two graphons give rise to the same invariant measure.

Theorem 2.18 ([Lov12, Theorem 13.10])**.** *For graphons U and W, the following are equivalent.*

(1) *$\mathbb{G}(\mathbb{N}, U)$ and $\mathbb{G}(\mathbb{N}, W)$ are the same distribution.*
(2) *$\delta_\square(U, W) = 0$.*
(3) *There are measure-preserving maps $\varphi, \psi \colon [0, 1] \to [0, 1]$ such that $U^\varphi = W^\psi$ a.e.*

When any of these equivalent conditions holds, we say that U and W are **weakly isomorphic***.*

We now describe a natural metric on the space of invariant measures.

Definition 2.19. Let $\mathcal{E}$ be the collection of extreme invariant measures, let $\mu \in \mathcal{E}$, and let F be a finite graph on $[n]$. Define $t_{\mathrm{ind}}(F, \mu) := \mu(\{G \in \mathcal{G} : G|_{[n]} = F\})$. Fix an enumeration $\langle F_i \rangle_{i \in \mathbb{N}}$ of finite graphs with underlying set $[n]$ for some $n \in \mathbb{N}$. For $\mu, \nu \in \mathcal{E}$, define

$$d_w(\mu, \nu) := \sum_{i \in \mathbb{N}} 2^{-i} \left| t_{\mathrm{ind}}(F_i, \mu) - t_{\mathrm{ind}}(F_i, \nu) \right|. \tag{7}$$

The following is standard.

Lemma 2.20. *The space $(\mathcal{E}, d_w)$ of extreme invariant measures is a compact Polish space with the topology of weak convergence. Further,*

$$\mathcal{S}_0^* := \left\{ \mathbb{G}(\mathbb{N}, G) : G \text{ is a finite graph} \right\} \tag{8}$$

is a dense subset.

Note that $\mathcal{S}_0^*$ also admits a straightforward computable enumeration.

The previous lemma tells us that we can approximate an extreme invariant measure arbitrarily well by measures which come from sampling graphons induced by finite graphs. A natural question is whether it is possible to take an invariant measure and find a (possibly random) sequence of finite graphs whose corresponding graphons almost surely converge to the invariant measure we started with. This is possible, as the following result states.

Lemma 2.21 ([Lov12, Corollary 11.15]). *Suppose U is a graphon. Then*

$$\big\langle \mathbb{G}\big(\mathbb{N}, \widehat{\mathbb{G}}(n, U)\big)\big\rangle_{n\in\mathbb{N}} \tag{9}$$

is a random sequence of extreme invariant measures that almost surely converges in $(\mathcal{E}, d_w)$ to $\mathbb{G}(\mathbb{N}, U)$.

3. Notions of computability for graphons and invariant measures on graphs

We begin by describing the notions of computability for graphons and invariant measures on graphs, and then present some of the basic relationships between them.

3.1. **Computable pseudometric spaces.** In order to describe computable elements of the spaces of graphons and invariant measures with respect to various pseudometrics, we will use the notion of a computable pseudometric space, a straightforward generalization of the notion of a computable metric space in computable analysis (see, e.g., [HR09]).

Definition 3.1. A **computable** (complete) pseudometric space consists of a triple $(M, d, \langle s_i\rangle_{i\in\mathbb{N}})$ such that

- (M, d) is a complete pseudometric space,
- $\langle s_i\rangle_{i\in\mathbb{N}}$ is dense in (M, d), and
- the sequence $\langle d(s_i, s_j)\rangle_{i<j\in\mathbb{N}}$ is a computable sequence of real numbers.

Definition 3.2. Suppose $(M, d, \langle s_i\rangle_{i\in\mathbb{N}})$ is a computable pseudometric space. A **rapidly converging Cauchy sequence** is a sequence $\langle s_{k_j}\rangle_{j\in\mathbb{N}}$ for which

$$d(s_{k_j}, s_{k_\ell}) \leq 2^{-j} \tag{10}$$

for $j < \ell \in \mathbb{N}$.

A rapidly converging Cauchy sequence is called a d-**name** for the limiting value $\lim_{n\to\infty} s_{k_n}$. We say that $\langle s_{k_j}\rangle_{j\in\mathbb{N}}$ is **computable** in d if the sequence of natural numbers $\langle k_j\rangle_{j\in\mathbb{N}}$ is computable, and that an element $s \in M$ is **computable** if it has some d-name that is computable. (These notions relativized to an oracle are defined in the obvious way.)

Roughly, a name for an element of the pseudometric space is a sequence of approximations that converges with rate $n \mapsto 2^{-n}$. Note that the choice of this rate is somewhat arbitrary, since given a sequence that converges with some other computable rate, one can computably "thin out" the sequence so that it converges at the rate we have chosen.

The computational strength needed to produce a d-name provides a measure of the complexity of the corresponding element of the represented space.

The pseudometric spaces we have considered so far can be straightforwardly made into computable pseudometric spaces using the computable enumerations of dense subsets we have identified.

Lemma 3.3. *The following are computable pseudometric spaces:*

- $(\mathcal{W}_0, d_1, \mathcal{S})$,
- $(\mathcal{W}_0, d_\square, \mathcal{S})$,
- $(\mathcal{W}_0, \delta_\square, \mathcal{S})$, *and*
- $(\mathcal{E}, d_w, \mathcal{S}_0^*)$.

In this paper we are interested in the relative computability of names for graphons and invariant measures considered as elements in these various computable pseudometric spaces.

3.2. **Computable relationships between representations.** In this section we want to consider the computable relationship between various representations of graphons and exchangeable arrays. In order to do this we need a notion of a *computable function* two pseudometric spaces.

Definition 3.4. Suppose (M, d, S) and (N, f, T) are computable pseudometric spaces. We say a map $g \colon M \to N$ is a **computable function**, or is simply **computable**, if there is a computer program e such that whenever $K := \langle k_j \rangle_{j \in \mathbb{N}}$ is an index sequence for a d-name of an element a then $\{e\}^{K,\ell}$ outputs an index set for an f-name of $g(a)$.

Suppose $h \colon N \to M$ is a computable map. We say that a computable map $g \colon M \to N$ is a **computable equivalence witnessed by** h if $d\big(x, h(g(x))\big) = 0$ for all $x \in M$ and $f\big(y, g(h(y))\big) = 0$ for all $y \in N$. In this case we say that the spaces are **computably equivalent**.

Let $k \colon M \to N$ be a surjective function. A computable map $h \colon N \to M$ is a **computable section** for k if $f\big(y, k(h(y))\big) = 0$ for all $y \in N$.

In other words, a function is computable if there is an algorithm that takes a name in one space and computably transforms it into a name in the other. A computable equivalence provides a uniform method for transforming a name in one space to a name in the other and vice-versa. Note that a computable equivalence induces a bijection between the corresponding metric spaces obtained by taking the quotient by distance 0 on each side.

We will consider computable sections in the case where the underlying sets M and N are the same and k is the identity function. In this case, a computable section takes an f-name for a computable element of N and returns a d-name for a (possibly different) computable element of M such that (N, f, T) cannot "distinguish" the points, in the sense that they have f-distance 0.

Consider the following notions for an invariant measure μ.

(1) There is a computable d_w-name for μ.
(2) There is a graphon W with a computable $\delta_\square$-name such that $\mathbb{G}(\mathbb{N}, W) = \mu$.
(3) There is a graphon W with a computable $d_\square$-name such that $\mathbb{G}(\mathbb{N}, W) = \mu$.
(4) There is a graphon W with a computable d_1-name such that $\mathbb{G}(\mathbb{N}, W) = \mu$.

The next theorem establishes relationships between these four notions which yield the implications in Corollary 3.6. In fact, as we will later see, these implications are all that are possible.

Theorem 3.5. *The following functions between pseudometric spaces are computable.*

(a) *A map $\alpha\colon (\mathcal{E}, d_w) \to (\mathcal{W}_0, \delta_\square)$ which takes $\mathbb{G}(\mathbb{N}, W)$ to some graphon U weakly isomorphic to W.*
(b) *The map $\beta\colon (\mathcal{W}_0, \delta_\square) \to (\mathcal{E}, d_w)$ which takes W to $\mathbb{G}(\mathbb{N}, W)$.*
(c) *The identity map $\mathrm{id}\colon (\mathcal{W}_0, d_\square) \to (\mathcal{W}_0, \delta_\square)$.*
(d) *The identity map $\mathrm{id}\colon (\mathcal{W}_0, d_1) \to (\mathcal{W}_0, d_\square)$.*

Furthermore, α is a computable equivalence witnessed by β, and vice-versa, and there is a computable section of the identity map (c). Finally, (a), (b), and (d) induce bijections on the corresponding metric spaces.

Proof. The d_1-distance between two graphons is at least their $d_\square$-distance, and so any d_1-name is a $d_\square$-name. Similarly, any $d_\square$-name is a $\delta_\square$-name. Hence (c) and (d) are computable.

Now to show (b) we want to show that if $\langle W_{G_n} \rangle_{n \in \mathbb{N}} \subseteq \mathcal{S}$ is a rapidly convergent Cauchy sequence in $\mathcal{S}$ then $\langle \mathbb{G}(\mathbb{N}, W_{G_n}) \rangle_{n \in \mathbb{N}}$ is a rapidly convergent sequence in $\mathcal{E}$. By the Counting Lemma ([Lov12, Exerise 10.30]), for any graphons U and V and finite graph F with k vertices, we have

$$\left| t_{\mathrm{ind}}(F, U) - t_{\mathrm{ind}}(F, V) \right| \leq 4\binom{k}{2} d_\square(U, V). \tag{11}$$

For any measure-preserving map $\varphi\colon [0,1] \to [0,1]$, we have $t_{\mathrm{ind}}(F, V) = t_{\mathrm{ind}}(F, V^\varphi)$. Therefore

$$\left| t_{\mathrm{ind}}(F, U) - t_{\mathrm{ind}}(F, V) \right| \leq 4\binom{k}{2} \delta_\square(U, V) \tag{12}$$

holds by Definition 2.10. Hence given a $\delta_\square$-name for U we can computably thin out its entries to form a d_w-name.

To show (a) we need the following Chernoff bound to obtain a representation in terms of $\delta_\square$ given the distribution of an exchangeable array. As shown in [Lov12, Lemma 10.16], for each $k \in \mathbb{N}$, with probability at least $1 - e^{-k/(2\log k)}$, we have

$$\delta_\square\big(U, W_{\widehat{\mathbb{G}}(k,U)}\big) \leq \frac{22}{\sqrt{\log k}}. \tag{13}$$

As $\mathbb{G}(k, U)$ depends only on the distribution of the induced exchangeable array we can find an element G_k which is within $\frac{44}{\sqrt{\log k}}$ in $\delta_\square$ of $W_{\widehat{\mathbb{G}}(k,U)}$ with probability at least $1 - e^{-k/(2\log k)}$, and so in particular

$$\delta_\square(G_k, U) \leq \frac{44}{\sqrt{\log k}}. \tag{14}$$

This therefore lets us create a sequence $\langle G_n \rangle_{k \in \mathbb{N}}$ such that $\langle W_{G_k} \rangle_{k \in \mathbb{N}}$ is a $\delta_\square$-name for any graphon with the same distribution as U.

To establish the computable equivalences in (a) and (b), we use the fact that convergence in $\delta_\square$ is equivalent to convergence of the corresponding random graphs (Theorem 2.18). This also shows that (a) and (b) induce bijections on the corresponding metric spaces.

We now show that (c) has a computable section. Assume we have a $\delta_\square$-name of U, and know a graph G_n such that

$$\delta_\square(W_{G_n}, U) < 2^{-(2^{2n}+1)}. \tag{15}$$

We will find a graph G_{n+1} such that

$$d_\square(W_{G_{n+1}}, W_{G_n}) \leq 45 \cdot 2^{-n} \tag{16}$$

and

$$\delta_\square(W_{G_{n+1}}, U) \leq 2^{-(2^{2(n+1)}+1)}. \tag{17}$$

This is enough to get a fast Cauchy sequence in $d_\square$. Find H such that

$$\delta_\square(W_H, U) < 2^{-(2^{2(n+1)}+1)}. \tag{18}$$

Then

$$\delta_\square(W_H, W_{G_n}) < 2^{-(2^{2n}+1)} + 2^{-(2^{2(n+1)}+1)} < 2^{-2^{2n}}. \tag{19}$$

There are graphs G'_n and H' both on the set $[|G_n| \cdot |H|]$ (where $|G_n|$ denotes the number of vertices of G_n, and similarly with H) such that

$$d_\square(W_{G'_n}, W_{G_n}) = d_\square(W_{H'}, W_H) = 0, \tag{20}$$

by taking blow-ups to a common refinement. Following the notation in [Lov12, §8.1.3], define the quantity

$$\widehat{\delta}_\square(G'_n, H') := \min_{\widehat{G'}_n, \widehat{H'}} d_\square(W_{\widehat{G'}_n}, W_{\widehat{H'}}), \tag{21}$$

where $\widehat{G'}_n$ ranges over the images of G'_n under permutations of $[|G_n| \cdot |H|]$, and similarly with $\widehat{H'}$. By [Lov12, Theorem 9.29], we have

$$\widehat{\delta}_\square(G'_n, H') \leq \frac{45}{\sqrt{-\log \delta_\square(W_{G'_n}, W_{H'})}} < \frac{45}{\sqrt{2^{2n}}} = 45 \cdot 2^{-n}. \tag{22}$$

Hence there is some reordering G_{n+1} of H' such that

$$d_\square(W_{G_{n+1}}, W_{G'_n}) \leq 45 \cdot 2^{-n}. \tag{23}$$

Because $d_\square(W_{G'_n}, W_{G_n}) = 0$, we have

$$d_\square(W_{G_{n+1}}, W_{G_n}) \leq 45 \cdot 2^{-n}. \tag{24}$$

By definition, $\delta_\square(W_{G_{n+1}}, W_{H'}) = 0$. Because $d_\square(W_{H'}, W_H) = 0$, we therefore have

$$\delta_\square(W_{G_{n+1}}, U) = \delta_\square(W_{G_{n+1}}, H) < 2^{-(2^{2(n+1)}+1)}, \tag{25}$$

as desired.

Finally, (d) induces a bijection on the corresponding metric spaces, as noted in Lemma 2.8. □

We have seen there there is a computable equivalence between $\delta_\square$-names for a graphon and names for the corresponding invariant measure. Further, given a $\delta_\square$-name, we can computably find a $d_\square$-name for a graphon yielding the same invariant measure. We have also seen that it is possible to transform a d_1-name to a $d_\square$-name in a computable way. It is therefore natural to ask whether there is a computable equivalence from a $d_\square$-name to a d_1-name. As we will see, in general there is not. This tells us that the d_1-name for a graphon contains fundamentally more computable information than an $d_\square$-name for a graphon.

As a consequence of Theorem 3.5, we obtain the following relationships among the numbered notions appearing after Definition 3.4.

Corollary 3.6. *For an invariant measure μ, notions (1), (2) and (3) are equivalent, and are all implied by notion (4).*

Our later results show that (4) is not implied by (1), (2), or (3).

4. $d_\square$-NAMES VS. d_1-NAMES: UPPER BOUND

Recall that a d_1-name is already a $d_\square$-name. In this section we establish that the halting problem $0'$ suffices as an oracle to computably transform a computable $d_\square$-name to a d_1-name. Further, in the random-free case, this oracle is not needed. In the next section we show that this is tight, in the sense that the use of $0'$ is necessary in general.

For $k \in \mathbb{N}$, let $\mathcal{P}_k$ denote the equipartition of $[0,1]$ into 2^k-many intervals of width 2^{-k}. For a graphon U we write $U_{\mathcal{P}_n}$ (as in [Lov12, §7.1]) to denote the step function graphon $\mathbb{E}[U \mid \mathcal{P}_n \times \mathcal{P}_n]$, i.e., the conditional expectation of the function U averaged on this $2^n \times 2^n$ square grid.

The following version of the weak regularity lemma for graphons follows immediately from [Lov12, Lemma 9.15 (a) and (b)] (taking $m = 1$ and $k = \lceil n/4 \rceil$ in (a) and then $m = 2^{\lceil n/4 \rceil}$ and $k = 2^n$ in (b), analogously to the proof of [Lov12, Lemma 10.16]).

Lemma 4.1 ([Lov12, Lemma 9.15]). *Let U be a graphon. Then for all n,*
$d_\square(U_{\mathcal{P}_n}, U) \leq 8/\sqrt{n}$.

Consider the space $\mathcal{M}$ of all martingales $\langle f_n \rangle_{n \in \mathbb{N}}$ where f_n is a $[0,1]$-valued step function graphon that is $\mathcal{P}_n \times \mathcal{P}_n$-measurable. Recall that being a martingale means that for every n, we have $\mathbb{E}[f_{n+1} \mid \mathcal{P}_n \times \mathcal{P}_n] = f_n$.

We endow this space with a natural topology which makes it effectively compact. In particular, we can view $\mathcal{M}$ as a closed subspace of the compact space $[0,1]^{\mathbb{N}}$ as follows. For each martingale $f = \langle f_n \rangle_{n \in \mathbb{N}}$ we define a corresponding element $x^f = \langle x_n^f \rangle_{n \in \mathbb{N}} \in [0,1]^{\mathbb{N}}$. Define the first coordinate x_0^f to be the value of the constant function f_0. Then define the next 4 coordinates $x_1^f, \ldots, x_4^f$ to be the values of the 2×2 step function f_1. Continue this way for each n, reading off 2^{n+1}-many coordinates from the $2^n \times 2^n$ step function f_n. Clearly this is an injection from $\mathcal{M}$ into $[0,1]^{\mathbb{N}}$. A martingale $f \in \mathcal{M}$ is said to be **computable** if the corresponding sequence $x^f \in [0,1]^{\mathbb{N}}$ is computable.

We say that an element of $[0,1]^{\mathbb{N}}$ encodes a martingale when it equals x^f for some martingale f. The subspace of $[0,1]^{\mathbb{N}}$ encoding a martingale is a Π_1^0 class, since if y does not code a martingale, then there is a computer program that, given y as an oracle, outputs this fact (by noticing that averaging fails at some level).

Given a graphon U that is computable in $d_\square$, we will find a Π_1^0 subclass of $\mathcal{M}$ that has a single element, which converges in d_1 to U. To establish the computability of this point, we will use the following lemma.

Lemma 4.2. *Suppose a Π_1^0 subset of $[0,1]^{\mathbb{N}}$ is a singleton. Then its unique member is computable.*

Proof. Let $T = \{\langle x_n \rangle_{n \in \mathbb{N}}\}$ be the singleton set. We can enumerate all the rational cylinder sets $[a_0, b_0] \times \cdots \times [a_{n-1}, b_{n-1}] \times [0, 1]^{\mathbb{N}}$ which are disjoint from the set T. We now describe how to compute the coordinate x_n for a given $n \in \mathbb{N}$.

Consider any rational numbers a and b such that $a < x_n < b$. By the compactness of $[0, 1]^{\mathbb{N}}$, we will eventually enumerate a finite rational cover of $[0, 1]^n \times [0, a] \times [0, 1]^{\mathbb{N}}$ and a finite rational cover of $[0, 1]^n \times [b, 1] \times [0, 1]^{\mathbb{N}}$ since these are compact sets disjoint from T. Moreover, since these enumerated covers are made of rational intervals, we will be able to computably determine when enough has been enumerated to cover the desired sets. By waiting for such covers to occur, we can computably learn that $a < x_n < b$.

This argument holds for all rationals a and b for which $a < x_n < b$, and so we can approximate x_n to arbitrary precision. $\square$

We now find the Π_1^0 subclass of $\mathcal{M}$ that has a single element.

Lemma 4.3. *Let U be a graphon computable in $d_\square$. Then the martingale $\langle U_{\mathcal{P}_n} \rangle_{n \in \mathbb{N}}$ is computable.*

Proof. Let $\mathcal{K}_U \subseteq \mathcal{M}$ be the collection of all martingales $\langle f_n \rangle_{n \in \mathbb{N}}$ satisfying $d_\square(f_n, U) \leq 8/\sqrt{n}$ for all $n \in \mathbb{N}$. Because U is computable in $d_\square$, $\mathcal{K}_U$ is a Π_1^0 class. Also note that $\mathcal{K}_U$ is nonempty since the martingale $U_{\mathcal{P}_n}$ is in $\mathcal{K}_U$ by Lemma 4.1. Further, every martingale in $\mathcal{K}_U$ converges in d_1 by the martingale convergence theorem (see, e.g., [Kal02, Theorem 7.23]) to a graphon V.

Now pick some $\langle f_n \rangle_{n \in \mathbb{N}} \in \mathcal{K}_U$. Since

$$d_\square(f_n, V) < d_1(f_n, V) \to 0 \tag{26}$$

and

$$d_\square(f_n, U) \leq 8/\sqrt{n} \to 0 \tag{27}$$

we have $d_\square(V, U) = 0$ and hence $U = V$ a.e. This shows that $\mathcal{K}_U$ only has one element, namely $U_{\mathcal{P}_n}$, and so $\langle f_n \rangle_{n \in \mathbb{N}} = U_{\mathcal{P}_n}$. Since $\mathcal{K}_U$ is a Π_1^0 class and has only one element, that element is computable by Lemma 4.2. $\square$

Note that this tells us that $U_{\mathcal{P}_n}$ is a computable martingale that converges in d_1, but it need not converge quickly. As we will see, in general we cannot computably identify a rapidly converging subsequence.

Theorem 4.4. *Let U be a graphon. Then from the jump of any $d_\square$-name of U, we can compute an d_1-name for it. In particular, if U has a computable $d_\square$-name, then it has a $0'$-computable d_1-name.*

Proof. Let X be some $d_\square$-name for U. Then using X we can compute a name for the martingale $U_{\mathcal{P}_n}$. Since $U_{\mathcal{P}_n}$ converges in d_1, its limit is computable in d_1

from the jump of a name for $U_{\mathcal{P}_n}$, which in turn is computable from the jump of X. □

It is natural to consider the case of random-free graphons, especially since the ability to flip between greyscale regions and black-and-white ones will be key to the lower-bound proof in Section 5.
In fact, in the random-free case, convergence of the martingale is tamer.

Lemma 4.5. *If $\langle f_n \rangle_{n \in \mathbb{N}}$ is a computable martingale that converges to a random-free graphon U, then U has a computable d_1-name.*

Proof. The key idea is that since U is 0–1-valued a.e., the value $d_1(f_n, U)$ is computable. This is because given a square in $\mathcal{P}_n \times \mathcal{P}_n$, the Lebesgue measure of those points (x, y) in the square such that $U(x, y) = 1$ is equal to the value that f_n takes there, and this suffices to compute the L_1 norm of the difference on that square. For example, for the constant function f_0, if $f_0 \equiv p$, then

$$d_1(f_0, U) = \lambda\{U = 0\} \cdot p + \lambda\{U = 1\} \cdot (1 - p)$$
$$= (1 - p)p + p(1 - p) = 2p(1 - p).$$

Since $\langle f_n \rangle_{n \in \mathbb{N}}$ is a d_1-name for U, and since we can compute each quantity $d_1(f_n, U)$, we may find a subsequence $\langle f_{k_n} \rangle_{n \in \mathbb{N}}$ such that $d_1(f_{k_n}, U) < 2^{-n}$. From this sequence $\langle f_{k_n} \rangle_{n \in \mathbb{N}}$ we can find a computable d_1-name for U. □

This implies that there is a computable procedure for translating d_1-names to $d_\square$-names in the case of a random-free graphon.

Theorem 4.6. *Let U be a random-free graphon computable in $d_\square$. Then U is computable in d_1.*

Proof. Since U is computable in $d_\square$, by Theorem 4.3 we can compute the martingale $\langle U_{\mathcal{P}_n} \rangle_{n \in \mathbb{N}}$, which converges to U in d_1. Then, because U is random-free, we can compute a d_1-name for the limit of this martingale by Lemma 4.5. □

We have just seen that for random-free graphons, unlike the general case, d_1 and $d_\square$-names can be computably transformed into each other. One might therefore wonder whether one can computably determine that a graphon is random-free. In fact, it is not possible to recognize when a graphon is random-free, as we now demonstrate. On the other hand $0'$ does allow us to recognize this.

Proposition 4.7. *The collection of d_1-names for random-free graphons is co-c.e. complete.*

Proof. Recall that $W(1 - W) \geq 0$ a.e. for a graphon W, as it takes values in $[0, 1]$. Also, W is random-free if and only if $\int W(1 - W) \, d\lambda = 0$. (See, e.g.,

[Jan13, Lemma 10.4].) Therefore, given a d_1-name for a graphon, we may compute $\int W(1-W)\,d\lambda$, and hence by noticing when this quantity is positive, we may enumerate the d_1-names of the non-random-free graphons. Hence the d_1-names of random-free graphons are co-computably enumerable.

We now show that the collection of d_1-names of random-free graphons is complete. For $s, e \in \mathbb{N}$, let U_e^s be the constant function 2^{-s} if $\{e\}_s(0)\uparrow$, and U_e^s be the constant function 2^{-k} if $k \leq s$ and minimal with $\{e\}_k(0)\downarrow$.

Observe that for each $e \in \mathbb{N}$, the sequence $\langle U_e^s \rangle_{s \in \mathbb{N}}$ is a computable d_1-name for a graphon that is not random-free if and only if $e \in 0'$. $\qquad\square$

In particular, there is no computer program that, given a d_1-name of a graphon, correctly asserts whether or not the graphon is random-free.

Having shown that every graphon with a computable $d_\square$-name has a d_1-name that is $0'$-computable, one may ask if this is tight, i.e., if $0'$ is necessary. We have just seen that this is not tight in the random-free case, and so any witness to the necessity of $0'$ must not be random-free. Next, in Section 5, we provide such an example.

5. $d_\square$-names vs. d_1-names: Lower bound

We have just seen that using $0'$ we can compute a d_1-name of a graphon given a computable $d_\square$-name for it. We now show that this is tight in the sense that there is a graphon that is computable in $d_\square$ such that $0'$ is computable from any d_1-name for a graphon weakly isomorphic to it. Furthermore, we may take this graphon to be a.e. continuous.

Theorem 5.1. *There is an a.e. continuous graphon U that is computable in $d_\square$, such that if V is weakly isomorphic to U then any d_1-name for V computes the halting problem $0'$.*

Proof. For each $n \in \mathbb{N}$, define the open interval $A_n := (1-2^{-n}, 1-2^{-(n+1)})$. The graphon U will take the value zero outside the block-diagonal $\bigcup_{n\in\mathbb{N}}(A_n \times A_n)$. Also, for each $n \in \mathbb{N}$, let $\ell_n := 1 - 2^{-(2n+1)}$, let $r_n := 1 - 2^{-(2n+2)}$, and let $m_n := \frac{\ell_n+r_n}{2}$, so that

$$0 < \ell_0 < m_0 < r_0 < \ell_1 < m_1 < r_1 < \ldots < 1. \tag{28}$$

Define the constant graphon $H^* \equiv \frac{1}{2}$. Because H^* is computable in $d_\square$, for each $s \in \mathbb{N}$ we can computably find a random-free step function graphon $G_s^* \in \mathcal{S}$ such that $\|H^* - G_s^*\|_\square < 2^{-s}$. For each $e \in \mathbb{N}$, let $\iota_e \colon [0,1] \to [\ell_e, r_e]$ be the unique increasing linear bijection. Note that $\iota_e(\frac{1}{2}) = m_e$. Let $G_{e,s} := \iota_e \circ G_s^*$ and $H_e := \iota_e \circ H^*$. Observe that

$$\|H_e - G_{e,s}\|_\square < \|H^* - G_s^*\|_\square < 2^{-s}. \tag{29}$$

Now for $s \in \mathbb{N}$, define U_s to be the graphon that is 0 outside of $\bigcup_{e \leq s}(A_e \times A_e)$ and for each $e \leq s$ is equal to the scaling to fit $A_e \times A_e$ of the following graphon $K_{e,s}$ on $[0,1] \times [0,1]$:

$$K_{e,s} := \begin{cases} H_e & \text{if } \{e\}_s(0)\!\uparrow, \text{ and} \\ G_{e,t} & \text{if } t \leq s \text{ is minimal such that } \{e\}_t(0)\!\downarrow . \end{cases} \tag{30}$$

Note that

$$\|U_s - U_{s+1}\|_\square \leq 2^{-s} + \sum_{n>s}\lambda(A_n \times A_n)$$
$$< 2^{-s} + 2^{-(s+1)} \cdot 2^{-(s+1)}$$
$$< 2^{-s+1}.$$

Clearly the graphons $K_{e,s}$ are uniformly computable in $d_\square$, and so the sequence $\langle U_s \rangle_{s \in \mathbb{N}}$ is a computable $d_\square$-name. For $x, y \in [0,1]$, define

$$U(x,y) = \lim_{s \to \infty} U_s(x,y) \tag{31}$$

when it is defined, and 0 otherwise. Note that U is a limit of the sequence $\langle U_s \rangle_{s \in \mathbb{N}}$, and it is a.e. continuous, as it is piecewise constant (i.e., a step function with countably many steps).

Note that for each $n \in \mathbb{N}$, we have

$$\lambda(U^{-1}\{\ell_n, m_n, r_n\}) = \lambda(A_n \times A_n) = 2^{-2(n+1)} \tag{32}$$

by construction, as each $K_{n,s}$ only takes values among ℓ_n, m_n, or r_n. For any graphon W define the set

$$X_W := \big\{x \in [0,1] \ : \ \lambda\big(W^{-1}(x)\big) > 0\big\}. \tag{33}$$

Note that for any V weakly isomorphic to U, by Theorem 2.18 condition (3) we have $X_V = X_U$. Further, as

$$\{\ell_n, m_n, r_n\} \cap \{\ell_p, m_p, r_p\} = \emptyset \tag{34}$$

for $p \neq n$, from a d_1-name for V we can compute the countable discrete set X_U. But $m_e \in X$ if and only if $\{e\}(0)\!\uparrow$, and so X_U computes $0'$. $\square$

6. Almost-everywhere continuity

In this section, we describe a random-free graphon that is computable in d_1 but not weakly isomorphic to any a.e. continuous graphon. Note that this is in contrast to the computable de Finetti theorem [FR12], which can be seen as saying that in a 1-dimensional analogue of this setting, the measurable object representing the sampler is a.e. computable, and in particular a.e. continuous. This provides another example of how the 2-dimensional case is considerably more complicated than the 1-dimensional case.

The notion of a.e. continuity is sensitive to the underlying topology of the space. Since a graphon is a function from $[0,1] \times [0,1]$ to $[0,1]$, it is reasonable to consider, as the topology on the domain, the product topology with respect to the usual topology on $[0,1]$. But there are situations where it is natural to consider graphons that are a.e. continuous with respect to other topologies on $[0,1]$ but are not weakly isomorphic to an a.e. continuous graphon with respect to the standard topology on $[0,1]$. We show that our result holds even in these more general situations, as long as the topology on $[0,1]$ still generates the Borel sets.

We begin by describing the construction of a random-free graphon G, which can be thought of as a symmetric measurable subset of $[0,1]^2$. We have drawn this measurable subset in Figure 1 as a black (1) and white (0) picture, with $(0,0)$ in the upper-left corner (similar to an adjacency matrix, and as is common when drawing graphons).

Construction. First, draw a 2×2 square grid (given by products of the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1]$ on each axis) and color the 2 squares on the diagonal black. Then on each of the 2 off-diagonal squares, draw a 4×4 square grid (similarly, from products of half-open or closed intervals) and color the 8 diagonal squares black. Then on each of the remaining 24 squares, draw an 8×8 grid and color the diagonal black. Continue in this way, filling diagonal squares within unfilled squares to obtain the graphon G.

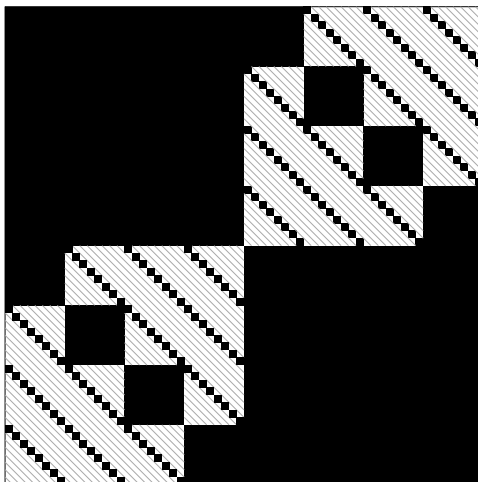This clearly describes a symmetric measurable subset of $[0,1]^2$, and hence a random-free graphon.

FIGURE 1. The graphon G, viewed as a subset of $[0,1]^2$.

The countable random graph induced by sampling from G may be thought of informally in the following way, which shows that it can be sampled in polynomial time: There is a questionnaire with an infinite list of questions indexed by positive integers. The nth question has 2^n possible answers. Each vertex corresponds to a person who has independently answered each question uniformly at random, independently from each other person. Two vertices are connected by an edge when the corresponding people agree on at least one answer to their questionnaires.

Lemma 6.1. *The graphon G constructed above is not a.e. continuous, has a computable d_1-name, and is twin-free.*

Proof. The black region $G^{-1}(\{1\})$ is clearly dense. Also the white region $G^{-1}(\{0\})$ has measure equal to $\alpha := \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots$ which is positive by the following calculation:

$$
-\log(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdots) = -\sum_{n=1}^{\infty} \log(1 - 2^{-n})
$$

$$
= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{2^{-kn}}{k}
$$

$$
= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} 2^{-kn}
$$

$$
= \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{2^k - 1} < \infty.
$$

So this graphon itself is not a.e. continuous, since $G^{-1}(\{0\})$ is a nowhere dense set of positive measure.

But G has a computable d_1-name since one can approximate the graphon in d_1 with some initial stage of the construction, as we now describe. Let G_n be the nth stage of the construction, and let $\beta_n := \int G_n \, d\lambda$. Then the measure of the remaining black area yet to be added to G is $(1 - \alpha) - \beta_n$, a computable real that rapidly tends to 0 as $n \to \infty$.

Finally, observe that G is twin-free, as by construction, each horizontal line gives rise to a different cross-section. $\square$

The main result of this section is that no graphon H weakly isomorphic to G is a.e. continuous (even with respect to other topologies that generate the Borel sets). The key combinatorial fact is the following.

Lemma 6.2. *Let G be the random-free graphon constructed above. Suppose $X, Y \subseteq [0, 1]$ are measurable sets such that $X \times Y$ is contained in $G^{-1}(\{0\})$ up to a nullset. Then $\lambda(X \times Y) = 0$.*

Proof. For each $n \in \mathbb{N}$, let X_n (respectively Y_n) be the union of all dyadic half-open intervals of size $2^{-(2^n-1)}$ whose intersection with X (respectively Y) has positive measure. To show that $\lambda(X \times Y) = 0$, we will show by induction that $\lambda(X_n \times Y_n) \leq 4^{-n}$.

The base case is trivial as $\lambda(X_0 \times Y_0) \leq 1$. For the induction step, consider each dyadic square $I \times J$ where $I \subseteq X_n$ and $J \subseteq Y_n$ are both of size $2^{-(2^n-1)}$. By construction, for each sub-dyadic interval $I' \subseteq I$ of size $2^{-(2^{n+1}-1)}$, there is a corresponding dyadic interval $J' \subseteq J$ of the same size such that $I' \times J'$ is a black square. If I' is disjoint from X (up to a nullset), then $I' \subseteq X_n \setminus X_{n+1}$. Otherwise, J' is disjoint from Y (up to a nullset), and $J' \subseteq Y_n \setminus Y_{n+1}$; for if not, then the black square $I' \times J'$ intersects $X \times Y$ outside a nullset, which cannot happen, since $X \times Y \subseteq G^{-1}(\{0\})$ is white. After considering all such sub-dyadic intervals I' we have that

$$\lambda(X_{n+1} \cap I) + \lambda(Y_{n+1} \cap J) \leq \frac{\lambda(I) + \lambda(J)}{2} = \lambda(I). \tag{35}$$

By the arithmetic–geometric mean inequality,

$$\lambda\big((X_{n+1} \times Y_{n+1}) \cap (I \times J)\big) = \lambda(X_{n+1} \cap I) \cdot \lambda(Y_{n+1} \cap J)$$
$$\leq \left(\frac{\lambda(X_{n+1} \cap I) + \lambda(Y_{n+1} \cap J)}{2} \right)^2$$
$$\leq \left(\frac{\lambda(I)}{2} \right)^2 = \frac{\lambda(I)^2}{4}.$$

Summing up over all such $I \times J$ and using the induction hypothesis we have

$$\lambda(X_{n+1} \times Y_{n+1}) \leq \frac{\lambda(X_n \times Y_n)}{4} \leq 4^{-(n+1)}. \tag{36}$$

Therefore $\lambda(X \times Y) = 0$. $\qquad \square$

We may now prove the main result about G.

Theorem 6.3. *Let G be the random-free graphon (which has a computable d_1-name) constructed above. Let H be a graphon weakly isomorphic to G, and let $\mathcal{T}$ be some topology on $[0,1]$ every open set of which is a standard Borel set. Then H is not a.e. continuous with respect to $\mathcal{T} \times \mathcal{T}$.*

Proof. Because G is twin-free and weakly isomorphic to H, by [Jan13, Theorem 8.6 (vi)] there is a measure-preserving map $\psi \colon [0,1] \to [0,1]$ such that $H = G^\psi$ a.e. Hence H must be random-free as well.

Now assume, towards a contradiction, that the map H is a.e. continuous with respect to $\mathcal{T} \times \mathcal{T}$. Because ψ is measure-preserving,

$$\lambda\big(G^{-1}(\{0\})\big) = \lambda\big(H^{-1}(\{0\})\big). \tag{37}$$

Define $A \subseteq [0,1]^2$ to be the set $H^{-1}(\{0\})$.

Since H is a.e. continuous with respect to $\mathcal{T} \times \mathcal{T}$, we have that A is a λ-continuity set, and therefore its interior (in the product topology $\mathcal{T} \times \mathcal{T}$) is standard Borel and has the same (positive) measure as A. Hence there is some open set $B \times C \subseteq A$ where B and C are $\mathcal{T}$-open sets of $[0,1]$ (and hence are standard Borel sets) that have positive measure.

Let $\lambda(\cdot|B)$ denote Lebesgue measure conditioned on B, that is

$$\lambda(A|B) = \frac{\lambda(A \cap B)}{\lambda(B)}. \tag{38}$$

Both $\lambda(\cdot|B)$ and $\lambda(\cdot|C)$ are well-defined since B and C have positive measure. Now, let μ_B and μ_C denote the pushforward measures on $[0,1]$ of $\lambda(\cdot|B)$ and $\lambda(\cdot|C)$ along the map ψ. That is, $\mu_B(S) = \lambda(\psi \in S|B)$ for all measurable $S \subseteq [0,1]$, and likewise with C. We claim that μ_B and μ_C are absolutely continuous with respect to λ. Indeed, if $\lambda(S) = 0$, then

$$\mu_B(S) = \lambda(\psi \in S|B) \leq \frac{\lambda(\psi \in S)}{\lambda(B)} = \frac{\lambda(S)}{\lambda(B)} = 0. \tag{39}$$

Therefore, the supports of μ_B and μ_A have positive λ-measure.

Because ψ is a measure-preserving map, we have

$$(\mu_B \otimes \mu_C)\{G = 0\} = \frac{1}{\lambda(A)\lambda(B)} \int_C \int_B \mathbf{1}_{\{G^\psi = 0\}} \, d\lambda \, d\lambda$$
$$= \frac{1}{\lambda(A)\lambda(B)} \int_C \int_B \mathbf{1}_{\{H = 0\}} \, d\lambda \, d\lambda$$
$$= 1,$$

where the last equality follows from the fact that $H = 0$ on $B \times C$.

Now let X be the support of μ_B and Y be the support of μ_C. Then $X \times Y$ is contained in $G^{-1}(\{0\})$ up to a nullset. By Lemma 6.2, we have $\lambda(X \times Y) = 0$, and so one of X and Y has measure 0, a contradiction to the fact that μ_B and μ_C are absolutely continuous probability measures. Hence H is not a.e. continuous with respect to $\mathcal{T} \times \mathcal{T}$. $\square$

We note that by taking the direct sum of this example with the lower bound construction of Section 5, we may obtain a $0'$-computable d_1-name for a graphon for which no weakly isomorphic graphon is a.e. continuous and from which any d_1-name computes $0'$. Namely, scale the graphon from Theorem 5.1 by $\frac{1}{2}$ and place it on $[0, \frac{1}{2}]^2$, along with G scaled by $\frac{1}{2}$ on $[\frac{1}{2}, 1]^2$.

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