Models with High Scott Rank

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Abstract

Scott rank is a measure of model-theoretic complexity; the *Scott rank* of a structure $\mathcal{A}$ in the language $\mathcal{L}$ is the least ordinal β for which $\mathcal{A}$ is prime in its $\mathcal{L}_{\omega\beta,\omega}$-theory. By a result of Nadel, the Scott rank of a structure $\mathcal{A}$ is at most $\omega_1^{\mathcal{A}}+1$, where $\omega_1^{\mathcal{A}}$ is the least ordinal not recursive in $\mathcal{A}$. We say that the Scott rank of $\mathcal{A}$ is *high* if it is at least $\omega_1^{\mathcal{A}}$. Let α be a Σ_1 admissible ordinal. A structure $\mathcal{A}$ of high Scott rank (and for which $\omega_1^{\mathcal{A}} = \alpha$) will have Scott rank $\alpha+1$ if it realizes a non-principal $\mathcal{L}_{\alpha,\omega}$-type, and Scott rank α otherwise.

For $\alpha = \omega_1^{CK}$, the least non-recursive ordinal, several sorts of constructions are known. The Harrison ordering $\omega_1^{CK}(1+\eta)$, where η is the order-type of the rationals, has Scott rank $\omega_1^{CK}+1$. Makkai constructs a model with Scott rank ω_1^{CK} whose $\mathcal{L}_{\omega_1^{CK},\omega}$-theory is $\aleph_0$-categorical. Millar and Sacks produce a model $\mathcal{A}$ with Scott rank ω_1^{CK} (in which $\omega_1^{\mathcal{A}} = \omega_1^{CK}$) but whose $\mathcal{L}_{\omega_1^{CK},\omega}$-theory is not $\aleph_0$-categorical.

We extend the result of Millar and Sacks to an arbitrary countable Σ_1 admissible ordinal α. For such α, we show that there is a model $\mathcal{A}$ with Scott rank α (in which $\omega_1^{\mathcal{A}} = \alpha$) whose $\mathcal{L}_{\alpha,\omega}$-theory is not $\aleph_0$-categorical.

When α is a Σ_1 admissible ordinal with $\omega_1 \leq \alpha < \omega_2$ we obtain a model with Scott rank α whose $\mathcal{L}_{\alpha,\omega}$-theory is not $\aleph_1$-categorical, but we are unable to preserve the admissibility of α within this structure.

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1 Introduction

Models which have elements of Scott rank unbounded below ω_1^{CK} often have an element of Scott rank ω_1^{CK}, in which case the model will have Scott rank at least $\omega_1^{\mathrm{CK}} + 1$. It is more difficult to avoid such an element while leaving open the possibility for other models of the theory at that level to realize such an element. Millar and Sacks [12] use a priority argument to construct a theory whose non-principal types may be omitted to give such a model. More specifically, they produce a model $\mathcal{A}$ with Scott rank ω_1^{CK} (in which $\omega_1^{\mathcal{A}} = \omega_1^{CK}$) but whose $\mathcal{L}_{\omega_1^{CK},\omega}$-theory is not $\aleph_0$-categorical. Here we extend their result to all countable admissible ordinals, and in a more limited way to admissible ordinals below ω_2.

In this first section, we provide background on the problem, an overview of our methods, and some basic results on Scott rank and in admissible recursion theory.

The theory is defined in Section 2, modulo the trees which will determine the types. We also show the theory to be complete and consistent (given appropriate conditions on the trees). The trees are defined by a priority argument in Section 3, and here we show them to satisfy these conditions.

We construct a model with the desired properties in Sections 4 and 5. In the countable case, we use Barwise compactness and type omitting. In the uncountable case, a somewhat weaker result follows using recent work of Sacks [17].

Sections 2, 3.2, and 4 largely follow work of Millar and Sacks [12].

1.1 Notation

For an ordinal β and set S, let $L(\beta, S)$ be the constructible universe relative to (the transitive closure of) S as an element, truncated at level β. Let ω_1^S be the least ordinal γ for which $L(\gamma, S)$ is Σ_1 admissible. We denote the cardinality of an ordinal α by $|\alpha|$.

Fix a structure $\mathcal{A}$ with an underlying first-order language $\mathcal{L}$. The infinitary extension $\mathcal{L}_{\infty,\omega}$ consists of formulas with arbitrary conjunctions and disjunctions (but still only finitely many universal and existential quantifiers), where we allow only finitely many free variables, but arbitrarily many constants. We also consider its restriction $\mathcal{L}_{\omega_1,\omega}$, which allows only countable conjunctions and disjunctions, and countably many constants. Set $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega} = \mathcal{L}_{\infty,\omega} \cap L(\omega_1^{\mathcal{A}}, \mathcal{A})$. Define $\mathcal{T}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ to be the complete theory of $\mathcal{A}$ in $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$. More details can be found in Barwise [2], Keisler [8], and Sacks [16].

1.2 Scott Rank

Scott [18] showed that when $\mathcal{A}$ is a countable structure in a countable language $\mathcal{L}$, it is characterized up to isomorphism among countable structures by a single sentence of $\mathcal{L}_{\omega_1,\omega}$, and in fact that there is a countable fragment $\mathcal{L}^{\mathcal{A}}$ of $\mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{A}$ is the atomic model of its complete theory in $\mathcal{L}^{\mathcal{A}}$. A similar result holds for higher cardinalities (see Barwise [2] VII.6.6). However, we will be concerned with the following result, which gives a more precise bound on Scott rank.

Nadel [13] later showed that $\mathcal{A}$ is a homogeneous model of $\mathcal{T}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ (in the fragment $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$). This holds for uncountable structures in a countable language as well (see, e.g., the argument in Chan [4] 1.13). Consider a type p over $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ which is realized in $\mathcal{A}$. Since p is first-order definable over $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$, the sentence $\bigwedge p$ is in the complete theory $\mathcal{T}'$ of $\mathcal{A}$ in $\mathcal{L}_{\omega_1,\omega} \cap L(\omega_1^{\mathcal{A}} + 1, \mathcal{A})$. Hence p becomes an atom of $\mathcal{T}'$ and so $\mathcal{A}$ is the atomic model of $\mathcal{T}'$.

We are interested in counting the depth of such infinitary conjunctions. We may define the Scott rank at once, from the top down.

Definition 1.1. *Let $\mathcal{A}$ be a structure in a countable language $\mathcal{L}$. We define the* Scott rank *of $\mathcal{A}$ to be the least ordinal β for which $\mathcal{A}$ is the prime model of its $\mathcal{L}_{\omega\beta,\omega}$-theory, which we call the* Scott theory *of $\mathcal{A}$.*

We may also give a characterization from the bottom up, making explicit the process of iteratively realizing types.

Definition 1.2. *We define languages $\mathcal{L}^{\mathcal{A}}_{\beta}$ by a Σ_1 recursion; for each ordinal β let $\mathcal{T}^{\mathcal{A}}_{\beta}$ be the complete $\mathcal{L}^{\mathcal{A}}_{\beta}$-theory of $\mathcal{A}$.*

Let $\mathcal{L}^{\mathcal{A}}_{1}$ be the first-order language $\mathcal{L}$ of $\mathcal{A}$. At limit ordinals take the union of the preceding languages, and at successor ordinals set $\mathcal{L}^{\mathcal{A}}_{\beta+1}$ to be the least fragment (closed under subformulas, conjunction, negation, and quantification) containing $\mathcal{L}^{\mathcal{A}}_{\beta} \cup \{\bigwedge p : p$ a non-principal type of $\mathcal{T}^{\mathcal{A}}_{\beta}\}$.

This also enables us to define the Scott rank of individual elements and tuples.

Definition 1.3. *The* Scott rank *of a tuple $\overline{a} \in \mathcal{A}^n$ is the least ordinal β for which the collection of formulas of $\mathcal{L}^{\mathcal{A}}_{\beta}$ in n free variables satisfied by $\overline{a}$ forms an orbit under automorphism. The* length of the Scott analysis *is the least ordinal β such that every tuple has Scott rank at most β.*

Lemma 1.4. *Suppose $\mathcal{T}$ is a Scott theory whose analysis has length β, and β is such that $\omega\beta = \beta$. If $\mathcal{A} \models \mathcal{T}$ then the Scott rank of $\mathcal{A}$ is at least β.*

Proof. See Millar-Sacks [12] 0.1. The key claim is that if $\mathcal{A}$ has Scott rank γ then the length of the Scott analysis is at most $\omega\gamma$. $\dashv$

Corollary 1.5. *Suppose $\mathcal{A}$ has atoms of Scott rank unbounded in $\omega_1^{\mathcal{A}}$. Then $\mathcal{A}$ has Scott rank $\omega_1^{\mathcal{A}}$ if it realizes no atom of Scott rank $\omega_1^{\mathcal{A}}$ (i.e., non-principal $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}}}$ type), and $\omega_1^{\mathcal{A}} + 1$ otherwise.*

Proof. Let $\beta = \omega_1^{\mathcal{A}}$ in Lemma 1.4. The Scott rank of $\mathcal{A}$ must be at least $\omega_1^{\mathcal{A}}$. If no non-principal types over $\mathcal{L}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ are realized, then the Scott rank of $\mathcal{A}$ is at most $\omega_1^{\mathcal{A}}$.

If a non-principal type is realized, the Scott rank of $\mathcal{A}$ is at least $\omega_1^{\mathcal{A}} + 1$. By Nadel's result and the observation which follows above, the Scott rank of $\mathcal{A}$ is at most $\omega_1^{\mathcal{A}} + 1$. $\dashv$

1.3 Examples

Harrison [7] showed that the linear ordering $\omega_1^{\mathrm{CK}}(1+\eta)$, where η is the order-type of the rationals, is recursively presentable (see, e.g., Ash-Knight [1] 8.11).

Using Nadel's result, one may show that an element of the Harrison ordering beyond the ω_1^{CK} initial segment is not definable by a recursive infinitary formula (and so has Scott rank equal to ω_1^{CK}); hence the entire structure has Scott rank $\omega_1^{\mathrm{CK}} + 1$. (For details, see Ash-Knight [1] 15.18.)

More generally, let $\alpha = \omega_1^X$ for any $X \subseteq \omega$. (By Sacks [14] there is such an X for any countable admissible ordinal $\alpha > \omega$.) Then there is an α-recursive linear ordering of type $\alpha(1+\eta)$ (see Keisler-Knight [9] 3.2.2). One may similarly show its Scott rank to be $\alpha + 1$.

It is easy to obtain a structure whose Scott rank equals ω_1^{CK}; merely take ω_1^{CK} itself as a linear ordering (and similarly with larger admissible ordinals). However, this structure is not even hyperarithmetical, nor does it preserve the admissibility of ω_1^{CK}. Furthermore, by Nadel [13] it has the same $\mathcal{L}_{\omega_1^{\mathrm{CK}},\omega}$-theory as the Harrison ordering. Since the latter is recursively presented and realizes all non-principal types, each non-principal type is $\Sigma_1^{\omega_1^{\mathrm{CK}}}$.

Makkai [11] later presented an arithmetical structure of Scott rank ω_1^{CK}. The

$\mathcal{L}_{\omega_1^{\mathrm{CK}},\omega}$-theory of this structure is $\aleph_0$-categorical. Knight and Millar [10] show how this construction can be made recursive, and with Calvert [3] present an $\aleph_0$-categorical recursive tree with Scott rank ω_1^{CK}.

More recently, Millar and Sacks [12] have presented a non-$\aleph_0$-categorical model $\mathcal{A}$ with Scott rank ω_1^{CK}, where $\omega_1^{\mathcal{A}} = \omega_1^{\mathrm{CK}}$ ("hyperarithmetically saturated", in the terminology of Knight-Millar [10]; we also say that $\mathcal{A}$ preserves the admissibility of ω_1^{CK}). Our results extend their methods to obtain similar structures with Scott rank α for countable Σ_1 admissible ordinals $\alpha > \omega_1^{\mathrm{CK}}$, and similar results (without $\omega_1^{\mathcal{A}} = \alpha$) for admissible $\alpha < \omega_2$.

1.4 Summary of Results

Let $\alpha < \omega_2$ be a Σ_1 admissible ordinal. We present a structure $\mathcal{A}$ such that $\mathcal{A}$ is an atomic model of $\mathcal{T}_\alpha^{\mathcal{A}}$ but $\mathcal{T}_\alpha^{\mathcal{A}}$ is not $|\alpha|$-categorical. Moreover, $\mathcal{T}_\alpha^{\mathcal{A}}$ is Δ_1^α (i.e., a Δ_1 subset of $L(\alpha, \mathcal{A})$) and no non-principal type is Σ_1^α. When α is countable, we may further require $\omega_1^{\mathcal{A}} = \alpha$.

First we will construct a theory $\mathcal{T}$ in a countable fragment of $\mathcal{L}_{\alpha,\omega}^{\mathcal{A}}$. In particular, $\mathcal{T}$ will have atoms of Scott rank unboundedly high in α, and will have countably many non-principal types, none of which are Σ_1^α.

We will show that $\mathcal{T}$ has an atomic model $\mathcal{A}$. Since $\mathcal{A}$ omits these non-principal types (of Scott rank α), but still has elements of rank unbounded in α, $\mathcal{A}$ has Scott rank α. We will see that $\mathcal{T}_\alpha^{\mathcal{A}}$ is not $|\alpha|$-categorical, as we can realize a non-principal type (producing a model with Scott rank at least $\alpha + 1$). When $\alpha < \omega_1$, we may further preserve the admissibility of α in the desired model.

1.5 Overview of the Construction

Our first goal is to obtain a complete and consistent Δ_1^α theory in $\mathcal{L}_{\alpha,\omega}$ with some, but only $|\alpha|$ many, non-principal types, none of which are Σ_1^α. Later we will use this theory to construct the desired model.

The n-types of our theory will be defined from specific trees $\{T_n^\delta : n < \omega,\ \delta < \alpha\}$, though we postpone their construction.

We begin by ensuring that we can consistently maintain a particular set of properties $\mathrm{TP}(\delta)$ of our trees at each level δ. These properties will enable the Scott analysis to extend through all levels. They also make the theory nearly have quantifier elimination; we use this to establish completeness.

We later use a priority argument to build trees satisfying these properties, while making all non-principal types non-Σ_1^α. The priority argument itself uses Lerman's tame Σ_2 approach to show that the injury sets are indeed α-finite.

To obtain the desired model $\mathcal{A}$ for countable α, we use Barwise compactness and effective type omitting. We present a Henkin argument which ensures that $\omega_1^{\mathcal{A}} = \alpha$ and that $\mathcal{A}$ does not realize the non-principal types of $\mathcal{T}_{\omega_1^{\mathcal{A}}}^{\mathcal{A}}$. By construction of the trees, $\mathcal{T}_{\omega_1^{\mathcal{A}}}^{\mathcal{A}}$ will not be $\aleph_0$-categorical.

When $\omega_1 \leq \alpha < \omega_2$, we use a result of Sacks [17] on models of size $\aleph_1$. Again we realize only the principal types of $T_\alpha^{\mathcal{A}}$, and so obtain a model of Scott rank α. We also show that there is a model of size $\aleph_1$ realizing a non-principal type, and so the theory is not $\aleph_1$-categorical.

1.6 α-Recursion Theory

In Millar-Sacks [12], the ω_1^{CK}-finite injury priority argument uses the Σ_1 projectum map $\pi : \omega_1^{\mathrm{CK}} \to \omega$ to reorder the requirements, thereby ensuring that the number of injuries to each requirement (indexed by $\beta \in \omega_1^{\mathrm{CK}}$) is actually finite (in fact, bounded by $2^{\pi(\beta)}$).

Here we work with an arbitary Σ_1 admissible α; in general $\sigma 1\mathrm{p}(\alpha) > \omega$. We will use Lerman's approach involving the tame Σ_2 projectum to give an α-finite bound on the injury to each requirement.

Most of the following will not be needed until the priority argument in Section 3, though we need a tame Σ_2 map into α with certain properties to define the theory in Section 2 and so we present it here.

We recall some definitions and results from Chong [5], Sacks [15], and Simpson [19].

Definition 1.6. *The Σ_1 projectum of α, written $\sigma 1\mathrm{p}(\alpha)$, is the least ordinal β for which there is an injective α-recursive map from α to β. We sometimes write α^* for $\sigma 1\mathrm{p}(\alpha)$.*

Lemma 1.7. *Suppose $A \subseteq \delta < \sigma 1\mathrm{p}(\alpha)$. If A is α-recursively enumerable, then A is α-finite.*

Proof. See Sacks [15] VII.2.1. $\dashv$

Lemma 1.8. *$\sigma 1\mathrm{p}(\alpha)$ is the least ordinal β for which there is a Σ_1^α definable subset of β which is not α-finite.*

Proof. See Sacks [15] VII.2.2. $\dashv$

Definition 1.9. *Let $\rho \leq \alpha$. The Σ_n^α cofinality of ρ, written $\sigma n\mathrm{cf}_\alpha(\rho)$, is the least $\gamma \leq \rho$ for which there is a Σ_n^α function mapping γ cofinally into ρ. We abbreviate $\sigma n\mathrm{cf}_\alpha(\alpha)$ as $\sigma n\mathrm{cf}(\alpha)$.*

Definition 1.10. *An α-cardinal is an ordinal $\gamma < \alpha$ which is a cardinal in the sense of α, i.e., there is no α-finite bijection between γ and any smaller ordinal. We define* $\mathrm{gc}(\alpha)$ *to be the greatest α-cardinal, if such exists, and α, otherwise.*

Definition 1.11. *A regular α-cardinal is an ordinal $\gamma < \alpha$ which is a regular cardinal in the sense of α, i.e., there is no α-finite map from any smaller ordinal to γ that has range unbounded in γ.*

Proposition 1.12 (Sacks-Simpson). *Let β be a regular α-cardinal and $\gamma < \beta$. Suppose $\{A_\delta : \delta < \gamma\}$ is a uniformly α-recursively enumerable set of α-finite subsets of α, each of α-cardinality less than β. Then $\cup\{A_\delta : \delta < \gamma\}$ is also α-finite and of α-cardinality less than β.*

Proof. See Sacks [15] VII.2.3. $\dashv$

Lemma 1.13. *If* $\sigma 1\mathrm{p}(\alpha) < \alpha$ *then* $\sigma 1\mathrm{p}(\alpha)$ *is the greatest α-cardinal.*

Proof. See Simpson [19] 0.12. $\dashv$

Corollary 1.14. $\mathrm{gc}(\alpha) \leq \alpha^*$.

Proof. By Lemma 1.13, either $\alpha^* = \alpha$, or $\alpha^* = \mathrm{gc}(\alpha)$. In either case, we have $\mathrm{gc}(\alpha) \leq \alpha^*$. $\dashv$

Definition 1.15. *A function $f : \gamma \to \alpha$ is tame Σ_2^α iff there is an α-recursive approximation that settles uniformly, i.e., an α-recursive function f' such that for all $\beta < \gamma$, there is a stage σ_0 such that*

$$f(x) = y \leftrightarrow (\forall \sigma > \sigma_0) f'(\sigma, x) = y$$

for all $x < \beta$.

Lemma 1.16. *Let f be a Σ_2^α function with* $\mathrm{dom}(f) \leq \sigma 2\mathrm{cf}(\alpha)$. *Then f is a tame Σ_2^α function.*

Proof. See Sacks [15] VIII.2.15. ⊣

Definition 1.17. $\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ *is the least ordinal β for which there is a tame Σ_2^α subset of β which is not α-finite.*

Lemma 1.18. $\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ *is the least ordinal β for which there is a tame Σ_2^α surjection of β onto α.*

Proof. See Chong [5] 1.59. ⊣

Corollary 1.19. $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \leq \alpha^*$.

Proof. See Sacks [15] VIII.2.4. ⊣

Lemma 1.20. $\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ *is the least ordinal β for which there is a tame Σ_2^α bijection from β to α.*

Proof. See Sacks [15] VIII.2.11. ⊣

Lemma 1.21. *If* $\mathrm{t}\sigma 2\mathrm{p}(\alpha) > \mathrm{gc}(\alpha)$ *then* $\mathrm{t}\sigma 2\mathrm{p}(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$.

Proof. See Sacks [15] VIII.2.5 or Chong [5] 1.60. ⊣

Definition 1.22. *Let t be a tame Σ_2^α bijection from* $\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ *to α (possible by Lemma 1.20). Let* $t^\sigma : \mathrm{t}\sigma 2\mathrm{p}(\alpha) \to \alpha$ *be an α-recursive approximation $t^\sigma(\xi) = t'(\sigma, \xi)$ as in Definition 1.15.*

Without loss of generality we may take $\{t^\sigma : \sigma < \alpha\}$ to be such that for $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, *the approximations converge up to ξ (though not necessarily correctly): $t^\xi \restriction (\xi + 1) \downarrow$. Further, for $\sigma < \alpha$ we may assume that $t^\sigma(\xi) \leq \sigma$ and that the* *approximations continuous, i.e., they only change values at successor stages. We also require that they don't change at successor of limit stages:*

$$t^{\omega\delta+1} \upharpoonright \omega\delta = t^{\omega\delta} \upharpoonright \omega\delta$$

for $\delta < \alpha$.

Finally, we fix a canonical enumeration of the α-finite sets and α-recursively enumerable sets. Let $k(\gamma, \nu)$ be an α-recursive function such that if $k(\gamma, \nu) = 0$ then $\gamma < \nu$, and for any α-finite set A, there is a $\gamma < \alpha$ for which $A = \{\gamma : k(\gamma, \nu) = 0\}$. For such A, write $A_\gamma := A$, so that $\{A_\gamma : \gamma < \alpha\}$ enumerates the α-finite sets.

Now let $r(\sigma, \varepsilon)$ be an α-recursively enumerable function such that if $\sigma \leq \tau$ then $A_{r(\sigma,\varepsilon)} \subseteq A_{r(\tau,\varepsilon)}$, and for any Σ_1^α set V, there is an $\varepsilon < \alpha$ for which $V = \bigcup\{A_{r(\sigma,\varepsilon)} : \sigma < \alpha\}$. For such V, write $V_\varepsilon := V$, so that $\{V_\varepsilon : \varepsilon < \alpha\}$ enumerates the α-recursively enumerable sets. Let $V_\varepsilon^\tau := A_{r(\tau,\varepsilon)}$ denote its approximation at stage τ.

2 Theory

2.1 Trees and Types

We will define a theory $\mathcal{T}_\delta$ and a property $\mathrm{TP}(\delta)$ of trees along with certain named branches, such that given trees $\{T_n^\delta : n < \omega\}$ with branches satisfying the property we can construct a Scott theory of height δ whose trees of partial types are the trees $\{T_n^\delta : n < \omega\}$.

The types will be of three sorts, defined in terms of three respective sorts of branches – regenerating branches, priority branches, and seed branches:

$$\{b_{n,k}^\delta : n < \omega,\ k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\}, \{q_{n,\gamma}^\delta : n < \omega,\ \gamma < \alpha\},\ \text{and}\ \{Q_n^\delta : n < \omega\},$$

respectively. The branch of the ξ^{th} potential candidate corresponding to a primary candidate branch s (isolated until such point as it becomes the primary candidate, if ever) is denoted by s_ξ.

Roughly speaking, the regenerating branches will keep the Scott analysis going, the priority branches will make the non-principal types non-Σ_1^α, and the seed branches ensure that we can keep making priority branches.

A tree T_n^δ will denote a subset of $2^{<\mathrm{t}\sigma 2\mathrm{p}(\alpha)\delta}$ that is closed under initial segments. The height of a branch s is denoted by $|s|$, and $s \restriction \gamma$ refers to the branch restricted to the first γ terms. We will later require that the string 000 is extended to a single branch, denoted Ω.

Let $\delta < \alpha$. To define the tree property $\mathrm{TP}(\delta)$ we introduce the following sets, which will be used for negative restrictions on the regenerating and priority types, respectively.

Definition 2.1. *Let $\beta < \delta$ and $n < \omega$. For each branch $s \in T_n^\beta$ we define the sets*

$$\begin{aligned}\mathrm{NegB}^\beta(s) &:= \{k : k < \mathrm{t}\sigma 2\mathrm{p}(\alpha) \textit{ and } s(\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta + k \cdot 2) = 0\}, \textit{ and} \\ \mathrm{tNegQ}_\gamma^\beta(s) &:= \{t^\gamma(k) : k < \mathrm{t}\sigma 2\mathrm{p}(\alpha) \textit{ and } s(\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta + k \cdot 2 + 1) = 0\},\end{aligned}$$

where t^β is our approximation to a bijection $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \to \alpha$ from Definition 1.22.

Definition 2.2. $\mathrm{TP}(\delta)$ *consists of the following statements.*

- *The trees cohere: For $\gamma < \delta$, $b_{n,k}^\delta$ extends $b_{n,k}^\gamma$, $q_{n,\beta}^\delta$ extends $q_{n,\beta}^\gamma$, and Q_n^δ extends Q_n^γ for all $n < \omega$, $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, and $\beta < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$. For $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma \leq \beta < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\delta$, $q_{n,\beta}^\delta$ extends Q_n^γ. Similarly, the potentially non-isolated branches cohere.*

- *The trees are continuously defined: For every limit ordinal $\omega\beta < \delta$, the limit $\lim_{\gamma < \omega\beta} b_{n,k}^\gamma$ exists and is equal to $b_{n,k}^{\omega\beta}$, and similarly for the other non-isolated branches and potentially non-isolated branches.*

- *For $s \in T_n^\beta$ and $\gamma < \beta \leq \delta$,*

$$\mathrm{NegB}^\gamma(s) \subseteq \mathrm{NegB}^\beta(s).$$

- *For $s \in T_n^\beta$ and $\gamma < \beta \leq \delta$,*

$$\mathrm{tNegQ}_\gamma^\gamma(s) \subseteq \mathrm{tNegQ}_\gamma^\beta(s).$$

- *The non-isolated branches and potentially non-isolated branches are 1 at all even heights, except that $q_{n,\gamma}^\beta(\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta + 2) = 0$ when $\gamma \leq \beta$.*

- *For every $\gamma < \delta$ and $n < \omega$ there are infinitely many branches $s \in T_n^\delta$ with $s(\gamma) = 1$.*

2.2 Languages

Suppose (for some $\delta < \alpha$) we are given trees $\{T_n^\delta : n < \omega\}$ which satisfy TP(δ). We now define the languages $\mathcal{L}_\beta$ and (in Section 2.3) the theories $\mathcal{T}_\beta$ for $\beta \leq \delta$.

The only actual non-logical function and relation symbols in the language will be a unary function f and the unary relations $\{S_i : i < \omega\}$. Let $\mathcal{L}_0$ be the first-order language with this signature. For convenience of notation, we will define *pseudo-predicates* $\{U_{n,\gamma} : n < \omega,\ \gamma < \alpha\}$ which will determine the relationship between formulas and branches in the trees. The languages $\mathcal{L}_\beta$ will be defined by induction in terms of these.

For each branch $s \in T_n^\delta$ and tuple $\overline{x}$ define the formula

$$\theta_{n,s}(\overline{x}) := \bigwedge_{\beta < |s|} (\neg)^{1+s(\beta)} U_{n,\beta}(\overline{x}).$$

We define the pseudo-predicates so that even subscripts correspond to the regenerating branches and odd subscripts correspond to priority branches (as suggested by our NegB and tNegQ definitions). (The seed branches will always make the positive choice $\exists y\ Q_{n+1}^\beta(\overline{x}, y)$ for relevant tuples $\overline{x}$.) For $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ define

$$\begin{aligned} U_{n,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta + k\cdot 2}(\overline{x}) &:= \exists y\ b_{n+1,k}^\beta(\overline{x}, y), \text{ and} \\ U_{n,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta + k\cdot 2+1}(\overline{x}) &:= \exists y\ q_{n+1,t^\beta(k)}^\beta(\overline{x}, y). \end{aligned}$$

The language $\mathcal{L}_{\beta+1}$ is the smallest expansion of

$$\mathcal{L}_\beta \cup \{U_{n,\gamma} : n < \omega,\ \gamma < \alpha\}$$

that is closed under conjunction, negation, and quantification. At limit ordinals simply take unions.

2.3 Theories

One can motivate the axioms by noting what is required in order to have a Scott analysis which continues through level α. For an illustration of how the axioms work at a simpler level, see the first two levels of the theory explicitly (along with corresponding consistency proofs) in Millar-Sacks [12].

The universal bootstrap (UB) axioms will ensure that an n-tuple $\overline{x}$ encodes information about the tree T_n^δ only if $f(x_{i+1}) = x_i$ for all $i \leq n$ and if x_1 is in the sort S_1. The S_i predicates will allow us to obtain something like quantifier elimination; f will map the sort S_{i+1} to S_i.

The universal tree (UT) axioms ensure that a branch of $2^{<\mathrm{t}\sigma 2\mathrm{p}(\alpha)\delta}$ is coded by an n-type only if the branch is in T_n^δ.

The existential closure (EC) axioms will be useful in obtaining the "pseudo-quantifier-free" normal form used to establish completeness.

Let $\beta \leq \delta$. We formally define $\mathcal{T}_\beta$ as the following collection of axioms.

UB: For all distinct $n, k < \omega$:

- $\forall x_1 ... \forall x_n(\ \neg\theta_{n,\Omega}(\overline{x}) \leftrightarrow (\ S_1(x_1) \wedge \bigwedge_{i<n}(f(x_{i+1}) = x_i)\ )\ )$
- $\forall x(\ S_1(x) \rightarrow (f(x) = x)\ )$
- $\forall x(\ \neg S_1(x) \rightarrow (f^n(x) \neq x)\ )$

- $\forall x(\ (f(x)=y) \rightarrow (S_{n+1}(x) \leftrightarrow S_n(y))\ )$

- $\forall x(\ S_n(x) \rightarrow \neg S_k(x)\ )$

- $\forall x\ \bigvee_{m<\omega} S_m(x)$

UT: For all $n<\omega$ and $s \in 2^{<\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta} \setminus T^{\beta}_{n}$:

- $\forall \overline{x}\ \neg\theta_{n,s}(\overline{x})$

EC: For all $n,k<\omega$:

(1) For all $u \in T^{1}_{n+1} \setminus \{\Omega\}$:

$$\forall \overline{x}\ \neg\theta_{n,\Omega}(\overline{x}) \rightarrow \exists^{>k} y\ \theta_{n+1,u}(\overline{x},y)$$

(2) Let $s \in T^{\beta}_{n} \setminus \{\Omega\}$ and $u \in T^{\beta}_{n+1} \setminus \{\Omega\}$ for which the following hold: If $u \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma = b^{\gamma}_{n+1,m}$ for a given $\gamma<\beta$ and $m<\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ then $s(\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + m\cdot 2)=1$. If $u \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma = q^{\gamma}_{n+1,m}$ for a given $\gamma<\beta$ and $m<\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ and $t^{\beta}(m)\downarrow$ then $s(\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + t^{\beta}(m)\cdot 2)=1$. Then:

$$\forall \overline{x}\ \theta_{n,s}(\overline{x}) \rightarrow \exists^{>k} y\ \theta_{n+1,u}(\overline{x},y)$$

(3) For $u \in T^{\beta}_{n+1}$:

$$(\exists y\ \theta_{n+1,u\restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta}(\overline{x}_n,y)) \rightarrow (\exists z\ \theta_{n+1,u}(\overline{x}_n,z))$$

(4) For all $\gamma<\beta$:

$$\forall \overline{x}_n\ \exists y(\ \neg\theta_{n,\Omega}(\overline{x}_n) \rightarrow Q^{\gamma}_{n+1}(\overline{x}_n,y)\ )$$

2.4 Consistency

Theorem 2.3. *The theory $\mathcal{T}_\delta$ determined by trees $\{T_n^\delta : n < \omega\}$ satisfying* TP(δ) *is consistent.*

Proof. We will build a model of $\mathcal{T}_\delta$ using infinite covers of the non-zero branches of $\{T_n^\delta : n < \omega\}$. We will use the fact that $\{T_n^\delta : n < \omega\}$ satisfy TP(δ) to show that the bootstrap, tree, and existential closure axioms are satisfied.

For each $n < \omega$ let M_n be an infinite-to-one cover of the non-zero branches of T_n^δ (disjoint from the other M_m's for $m \neq n$), and let s be a map witnessing this:

$$s : \bigcup M_n \to \{\text{branches of } T_n^\delta\}.$$

Now set

$$M := \bigcup_{n<\omega} \prod_{i \leq n} M_i.$$

Denote an element $(a_1, a_2, ..., a_n)$ by $(\overline{a}_n)$, and write $s[a_n] \in T_n^\delta$ for the value of s on input $a_n \in M_n$. Let $s[a_n](\gamma)$ denote the γ^{th} bit of $s[a]$. Define the sets

$$\begin{aligned} \text{NEGB} \;&:=\; \{\overline{a}_n \in \mathcal{M} : \exists i < n, \;\; k < \text{t}\sigma\text{2p}(\alpha), \;\; \varepsilon < \beta \;\; \text{s.t.} \\ &\qquad a_{i+1} \restriction \text{t}\sigma\text{2p}(\alpha)\varepsilon = b^{\varepsilon}_{i+1,k} \text{ and } k \in \text{NegB}^{\varepsilon}(s[a_i])\}. \end{aligned}$$

and

$$\begin{aligned} \text{NEGQ} \;&:=\; \{\overline{a}_n \in \mathcal{M} : \exists i < n, \;\; k < \text{t}\sigma\text{2p}(\alpha), \;\; \varepsilon < \beta, \;\; \text{s.t. } t^{\varepsilon}(k)\downarrow, \\ &\qquad a_{i+1} \restriction \text{t}\sigma\text{2p}(\alpha)\varepsilon = q^{\varepsilon}_{i+1,t^{\varepsilon}(k)}, \text{ and } t^{\varepsilon}(k) \in \text{tNegQ}^{\varepsilon}_{\varepsilon}(s[a_i])\}. \end{aligned}$$

The universe of our model will be $\mathcal{M} := M \backslash \text{NEGB} \backslash \text{NEGQ}$. We now determine the atomic diagram (i.e., where f and the S_i's hold). Define $f((\overline{a}_n)) = (\overline{a}_{n-1})$; this is well-defined as $\mathcal{M}$ is closed under initial subterms. Let $S_k((\overline{a}_n))$ hold iff $k = n$. We let the basic partial type of the n-tuple $((\overline{a}_1), (\overline{a}_2), ..., (\overline{a}_n))$ be $\theta_{n,s(a_n)\restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)}$.

We now show that the branch $s[a_n]$ encodes the basic partial $\mathcal{L}_{\delta+1}$ type of $((\overline{a}_1), (\overline{a}_2), ..., (\overline{a}_n)) \in \mathcal{M}^n$. Let $\gamma < \beta$. We must show that $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + k) = 1$ iff $U_{n,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma+k}((\overline{a}_n), ..., ((\overline{a}_n))$ holds. If $\gamma = 0$ then $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + k = k$ and we are done as we have defined the atomic diagram so that $s[a_n](k) = 1$ iff $U_{n,k}((\overline{a}_n), ..., ((\overline{a}_n))$.

Suppose $\gamma \geq 1$; we deal first with even branch lengths. Assume $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$ $+k \cdot 2) = 1$. By definition of M_{n+1} there is some element $a_{n+1} \in M_{n+1}$ for which $s(a_{n+1} \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma) = b^{\delta}_{n+1,k}$. The tree property $\mathrm{TP}(\delta)$ tells us that for all $\varepsilon < \gamma$ we have $\mathrm{NegB}^{\varepsilon}(s[a_n]) \subseteq \mathrm{NegB}^{\gamma}(s[a_n])$. Therefore $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon + k \cdot 2) = 1$ for all $\varepsilon < \gamma$ and so $(\overline{a}_{n+1}) \notin \mathrm{NEGB}$.

We now show that $(\overline{a}_{n+1}) \notin \mathrm{NEGQ}$ and which will tell us that $(\overline{a}_{n+1}) \in \mathcal{M}$. Now either for all $\varepsilon < \gamma$ we have $b^{\delta}_{n+1,k} \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon = b^{\gamma}_{n+1,k}$ or else $b^{\delta}_{n+1,k} \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma = (b^{\gamma}_{n+1,k})_{\xi}$, a potential candidate (which is still isolated in T^{ε}_{n+1}). Either possibility implies that $b^{\gamma}_{n+1,k} \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon \neq q^{\varepsilon}_{n+1,\lambda}$ for any λ, and so $(\overline{a}_{n+1}) \notin \mathrm{NEGQ}$.

Suppose $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + k \cdot 2) = 0$. If there were an element $c \in \mathcal{M}$ for which $\theta_{n+1,b^{\gamma}_{n+1,k}}((\overline{a}_1), ..., (\overline{a}_n), c)$ held then there would be an element $a_{n+1} \in M_{n+1}$ for which $c = (\overline{a}_{n+1}) \in \mathcal{M}$ and $s[a_{n+1}] \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon = b^{\gamma}_{n+1,k}$ for $\varepsilon < \gamma$. But as $k \cdot 2 \in \mathrm{NegB}^{\gamma}(s[a_n])$, we have $(\overline{a}_{n+1}) \in \mathrm{NEGB}$ and so $\overline{a}_{n+1} \notin \mathcal{M}$, a contradiction.

Suppose $\gamma \geq 1$; we now consider odd branch lengths. Assume $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + k \cdot 2 + 1) = 1$. By definition of M_{n+1} there is some element $a_{n+1} \in M_{n+1}$ for which $s(a_{n+1} \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma) = q^{\delta}_{n+1,t^{\delta}(k)}$. The tree property $\mathrm{TP}(\delta)$ tells us that for all $\varepsilon < \gamma$ we have $\mathrm{tNegQ}^{\varepsilon}_{\varepsilon}(s[a_n]) \subseteq \mathrm{tNegQ}^{\gamma}_{\varepsilon}(s[a_n])$. Therefore $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon + k \cdot 2 + 1) = 1$

for all $\varepsilon < \gamma$ and so $(\overline{a}_{n+1}) \notin$ NEGQ.

We now show that $(\overline{a}_{n+1}) \notin$ NEGB and which will tell us that $(\overline{a}_{n+1}) \in \mathcal{M}$. Now either for all $\varepsilon < \gamma$ we have $q^{\delta}_{n+1,t^{\delta}(k)} \upharpoonright \mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon = q^{\gamma}_{n+1,t^{\delta}(k)}$ or else $q^{\delta}_{n+1,t^{\delta}(k)} \upharpoonright \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma = (q^{\gamma}_{n+1,t^{\delta}(k)})_{\xi}$, a potential candidate (which is still isolated in T^{ε}_{n+1}). Recalling our conditions on the approximations t^{γ}, either possibility implies that $q^{\gamma}_{n+1,t^{\gamma}(k)} \upharpoonright \mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon \neq b^{\varepsilon}_{n+1,\lambda}$ for any λ, and so $(\overline{a}_{n+1}) \notin$ NEGB.

Now suppose $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + k \cdot 2 + 1) = 0$. If there were an element $c \in \mathcal{M}$ for which $\theta_{n+1,q^{\gamma}_{n+1,t^{\gamma}(k)}}((\overline{a}_1), ..., (\overline{a}_n), c)$ held then there would be an element $a_{n+1} \in M_{n+1}$ for which $c = (\overline{a}_{n+1}) \in \mathcal{M}$ and $s[a_{n+1}] \upharpoonright \mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon = b^{\gamma}_{n+1,k}$ for $\varepsilon < \gamma$. But as $k \cdot 2 \in \mathrm{tNegQ}^{\gamma}_{\gamma}(s[a_n])$, we have $(\overline{a}_{n+1}) \in$ NEGQ and so $\overline{a}_{n+1} \notin \mathcal{M}$, a contradiction.

We now show that $\mathcal{M} \models \mathcal{T}_{\delta}$. The axioms (UB) are satisfied by our choice of atomic diagram, which maps each S_{i+1} into S_i via f, satisfying the required conditions.

Let $\gamma < \delta$. To show (UT), consider sequences $((\overline{a}_1), ..., (\overline{a}_n))$ for $a_i \in M_i$ where $(\overline{a}_i) \in \mathcal{M}$ (for all $i \leq n$) and branches $u \notin T^{\gamma}_n$ with $|u| < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$. (In all other cases of $\overline{a} \in \mathcal{M}$, we have $\theta_{n,\Omega}(\overline{a})$.) But then there is some $\varepsilon < \gamma$ for which $s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon) \neq u(\mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon)$. Since its partial type is determined by the branch, as shown above, we have that $U^{s[a_n](\mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon)+1}_{n,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\varepsilon}((\overline{a}_1), ..., (\overline{a}_n))$ holds, and so $\neg\theta_{n,u}((\overline{a}_1), ..., (\overline{a}_n))$ as desired.

We now show (EC) axiom (1). Let $u \in T^{\gamma}_{n+1} \setminus \{\Omega\}$ with $|u| < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, and let $d_1, ..., d_n \in \mathcal{M}$ be such that $\neg\theta_{n,\Omega}(d_1, ..., d_n)$ holds. By (UT) there is a branch $s \in T^{\gamma}_n$ of length $|s| = \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$ for which $\theta_{n,s}(d_1, ..., d_n)$ holds. By our construction, there must be a sequence $a_i \in M_i$ for $i \leq n$ with $s[a_n] = s$ and for which each $d_i = (a_1 \cdot ... \cdot a_i)$. There is a branch v^{γ}_{n+1} extending u but not extending any of the non-isolated branches $b^1_{n+1,k}$ or $q^1_{n+1,t^{\gamma}(k)}$ for $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$.

For each $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ choose $a_{n+1,k} \in M_{n+1}$ so that $s[a_{n,k}] = v$. Set $c_k := (a_1 \cdot ... \cdot a_n \cdot a_{n+1,k}) \in M \setminus \mathrm{NEGB} \setminus \mathrm{NEGQ}$. We thus have infinitely many terms $c_k \in \mathcal{M}$ which satisfy $\theta_{n+1,u}(d_1, ..., d_n, c_k)$.

To show (EC) axiom (2), let $s \in T^\gamma_n \setminus \{\Omega\}$ and $u \in T^\gamma_{n+1} \setminus \{\Omega\}$ satisfying the hypotheses of the axiom, and let $d_1, ..., d_n \in \mathcal{M}$ be such that $\theta_{n,s}(d_1, ..., d_n)$ holds. By our construction, according to our hypothesis there must be a sequence $a_i \in M_i$ for $i \leq n$ with $s[a_n] = s$ and for which each $d_i = (a_i \cdot ... \cdot a_i)$. There is a branch v^γ_{n+1} extending u but not extending any of the non-isolated branches $b^\gamma_{n+1,k}$ or $q^\gamma_{n+1,t^\gamma(k)}$ for $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$. For each $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ choose $a_{n+1,k} \in M_{n+1}$ so that $s[a_{n,k}] = v$. Set $c_k := (a_1 \cdot ... \cdot a_n \cdot a_{n+1,k}) \in M \setminus \mathrm{NEGB} \setminus \mathrm{NEGQ}$. We thus have infinitely many terms $c_k \in \mathcal{M}$ which satisfy $\theta_{n+1,u}(d_1, ..., d_n, c_k)$.

For (EC) axiom (3), consider a branch $u \in T^\gamma_{n+1}$ and elements $d_1, ..., d_n \in \mathcal{M}$ for which there is an element $d_{n+1} \in \mathcal{M}$ such that $\theta_{n+1,u\restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma}(d_1, ..., d_n, d_{n+1})$ holds. By our construction, there must be a sequence $a_i \in M_i$ for $i \leq n$ with $u[a_n] = u$ and for which each $d_i = (a_i \cdot ... \cdot a_i)$. Because there is an extension to $n+1$ variables at level $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$, by $\mathrm{TP}(\delta)$ there is a branch v^γ_{n+1} extending u but not extending any of the non-isolated branches $b^\gamma_{n+1,k}$ or $q^\gamma_{n+1,t^\gamma(k)}$ for $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$. Then choose $a_{n+1} \in M_{n+1}$ so that $s[a_{n,k}] = v$. Set $c := (a_1 \cdot ... \cdot a_n \cdot a_{n+1}) \in M \setminus \mathrm{NEGB} \setminus \mathrm{NEGQ}$. We thus obtain an element $c \in \mathcal{M}$ which satisfies $\theta_{n+1,u}(d_1, ..., d_n, c)$.

Finally, we show (EC) axiom (4). Let $d_1, ..., d_n \in \mathcal{M}$ be such that the formula $\neg\theta_{n,\Omega}(d_1, ..., d_n)$ holds. By (UT) there is a branch $s \in T^\gamma_n$ of length $|s| = \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$ for which $\theta_{n,s}(d_1, ..., d_n)$ holds. As before, there is a sequence $a_i \in M_i$ for $i \leq n$ with $s[a_n] = s$ and with each $d_i = (a_1 \cdot ... \cdot a_i)$. By $\mathrm{TP}(\delta)$, there is a seed branch Q^γ_{n+1} extending s but not extending any of the non-isolated branches $b^\gamma_{n+1,k}$ or $q^\gamma_{n+1,t^\gamma(k)}$ for $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$. Let $a_{n+1} \in M_{n+1}$ so that $s[a_n] = Q^\gamma_{n+1}$. Set $c := (a_1 \cdot ... \cdot a_n \cdot a_{n+1}) \in M \setminus \mathrm{NEGB} \setminus \mathrm{NEGQ} = \mathcal{M}$, satisfying $Q^\gamma_{n+1}(d_1, ..., d_n, c)$. $\dashv$

2.5 Completeness

Definition 2.4. *Let $\beta+1<\delta$ be a successor ordinal. We define the language of* pseudo-quantifier-free rank $\beta+1$ formulas *to be the closure under negation and conjunction of*

$$\mathcal{L}_\beta \cup \{U_{n,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta+k}(\overline{x}_n) : n<\omega,\ k<\mathrm{t}\sigma 2\mathrm{p}(\alpha)\} \cup \{\exists x \bigwedge_{i<\omega} \neg S_i(x)\}.$$

At limit ordinals $\omega\gamma \le \delta$, the pseudo-quantifier-free rank $\omega\gamma$ formulas are all those in $\mathcal{L}_{\omega\gamma}$.

We now describe a normal form for formulas of $\mathcal{L}_\beta$. For a finite increasing sequence of finite integers $\sigma = \sigma(1)...\sigma(|\sigma|)$, define the sequence of strings $\overline{s}_\sigma$ to be $s_{\sigma(1)}...s_{\sigma(|\sigma|)}$, where for $1 \le i \le |\sigma|$ each $s_{\sigma(i)}$ is some branch through $T^\beta_{\sigma(i)}$. Then for all $\gamma < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta$ define the restriction of the sequence $\overline{s}_\sigma \restriction \gamma$ to be the sequence of the restrictions $s_{\sigma(1)\restriction\gamma}...s_{\sigma(|\sigma|)\restriction\gamma}$. Now set

$$\Delta^\gamma[\overline{s}_\sigma](x) := \bigwedge_{i\le|\sigma|} \theta_{i,s_{\sigma(i)}\restriction\gamma}(f^{n-1}(x),...,f^{n-\sigma(i)}(x)).$$

For an irreflexive directed graph G on the set $\{1,...,n\}$ with each vertex the source of exactly one edge, set

$$\varphi_G(\overline{x}_n) := \bigwedge_{(i\to j)\in\mathrm{Edge}(G)} f(x_i)=x_j, \text{ and}$$

$$\Delta^0_{G,S}(\overline{x}_n) := \varphi_G(\overline{x}_n)\wedge\varphi_S(\overline{x}_n),$$

where φ_S is a conjunction in the language $\{S_i : i<\omega\}$.

Definition 2.5. *A* basic formula of rank γ *is any formula*

$$\Delta^0_{G,S}(\overline{x}_n) \wedge \bigwedge_{b \text{ a source vertex of } G} \Delta^\gamma[\overline{s}_{\sigma_b}](x_b)$$

where G is an irreflexive graph on $\{1, ..., n\}$ with constant out-degree one and S and σ are as above.

If φ is a basic formula of rank γ and $\gamma < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta$ then $\varphi \in \mathcal{L}_\beta$.

Lemma 2.6. *Let $\beta \leq \delta$, and suppose $\varphi(\overline{x}, y)$ is a pseudo-quantifier-free basic formula of $\mathcal{L}_\beta$. Then there is a pseudo-quantifier-free formula $\zeta_\varphi(\overline{x})$ for which $\mathcal{T}_\beta \vdash \forall \overline{x}((\exists y \varphi(\overline{x}, y)) \leftrightarrow \zeta_\varphi(\overline{x}))$.*

Proof. For $\beta + 1 < \delta$ a successor ordinal, consider a pseudo-quantifier-free formula $\varphi(\overline{x}, y)$ of $\mathcal{L}_{\beta+1}$, and consider the basic formulas of rank $< \mathrm{t}\sigma 2\mathrm{p}(\alpha)(\beta + 1)$ which describe the branches of $\{T^{\beta+1}_n : n < \omega\}$ that extend its restriction to level $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta$. $\mathrm{TP}(\delta)$ ensures that a branch at level $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta$ has countably many extensions to level $\mathrm{t}\sigma 2\mathrm{p}(\alpha)(\beta + 1)$. The theory $\mathcal{T}_{\beta+1}$ proves that φ is equivalent to the countable disjunct of of these basic formulas. By the following Lemma 2.7, we have quantifier elimination in $\mathcal{T}_{\beta+1}$ for each of these basic formulas. Then $\exists y \varphi(\overline{x}, y)$ is equivalent in $\mathcal{T}_{\beta+1}$ to the disjunct of the corresponding quantifier-free equivalents. It is pseudo-quantifier-free as we are allowed formulas involving all of $\{U_{n,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta}(\overline{x}_n) : n < \omega\}$.

The result for limit ordinals $\omega\gamma \leq \delta$ is clear from the definition of pseudo-quantifier-free formulas at limit levels. $\dashv$

Lemma 2.7. *Let $\beta + 1 < \delta$, and suppose $\varphi(\overline{x}, y)$ is a basic formula of rank $< \mathrm{t}\sigma 2\mathrm{p}(\alpha)(\beta + 1)$. Then there is a pseudo-quantifier-free formula $\zeta_\varphi(\overline{x})$ for which $\mathcal{T}_{\beta+1} \vdash \forall \overline{x}((\exists y \varphi(\overline{x}, y)) \leftrightarrow \zeta_\varphi(\overline{x}))$.*

Proof. Let $\beta + 1 < \delta$ be a successor ordinal. By simultaneous induction (on the present lemma and Lemma 2.6) we may assume that we have elimination of quantifiers for pseudo-quantifier-free formulas of $\mathcal{L}_\gamma$ where $\gamma \leq \beta$. Without loss of generality, we may assume that $\overline{x}, y$ is closed under f. Then we know that y is a source vertex and that $\varphi(\overline{x}, y)$ has one of the following two forms:

$$\theta_{1,s}(y) \wedge \xi(\overline{x}), \text{ or}$$

$$\Delta^\gamma[\overline{s}_\sigma](y) \wedge (f(y) = x) \wedge \xi(\overline{x}),$$

where $\xi(\overline{x})$ is a basic formula which does not involve y.

In the first case, consider whether or not $s \in T_1^\beta$. If $s \in T_1^\beta$ then by (EC) axiom (2) the theory $\mathcal{T}_\beta$ proves that there are infinitely many z for which $\theta_{1,s}(z)$ holds. We may rearrange the terms so that $\xi(\overline{x})$ is equivalent to $\bigwedge_{i \in I} \theta_{1,s}(x_i) \wedge \Delta^\gamma(\overline{x})$ for some finite set I, and where Δ^γ is a formula that does not involve $\theta_{1,s}$. But then (EC) axiom (2) (for $k > |I|$) tells us that $\mathcal{T}_{\beta+1} \vdash (\exists y\ \theta_{1,s}(y) \wedge \bigwedge_{i \in I} \theta_{1,s}(x_i)) \leftrightarrow (\bigwedge_{i \in I} \theta_{1,s}(x_i))$. Let $\xi_\varphi := (\bigwedge_{i \in I} \theta_{1,s}(x_i)) \wedge \Delta^\gamma$. Then $\mathcal{T}_{\beta+1} \vdash (\exists y\ \varphi(\overline{x}, y) \leftrightarrow \xi_\varphi(\overline{x})$.

If $s \notin T_1^\beta$ then the (UT) axioms let us choose the desired $\xi_\varphi(\overline{x})$ to be false.

In the latter case, let $n := \sigma(|\sigma|)$. Then the n-tuple $(f^{n-1}(y), ..., f(y), y) = (f^{n-2}(x), ..., x, y)$, and for $i \neq |\sigma|$ we have the n-tuple $(f^{n-1}(y), ..., f^{n-\sigma(i)}(y)) = (f^{n-2}(x), ..., f^{n-\sigma(i)-1}(x))$. Expanding $\Delta^\gamma[\overline{s}_\sigma](y)$ according to this we obtain

$$\Delta^\gamma[\overline{s}_\sigma](y) = \theta_{n,s_n\restriction\gamma}(f^{n-2}(x), ..., x, y) \wedge \bigwedge_{i<|\sigma|} \theta_{\sigma(i),s_{\sigma(i)}\restriction\gamma}(f^{n-2}(x), ..., f^{n-\sigma(i)-1}(x)).$$

By the (UB) axioms, $\mathcal{T}_\beta \vdash \theta_{n,s_n\restriction\gamma}(f^{n-2}(x), ..., x, y) \to (f(y) = x)$, and so we may drop the clause $f(y) = x$. Leaving those subterms of Δ^γ involving x, we see that

$\varphi(\overline{x}, y)$ is equivalent to the formula

$$\theta_{n,u}(f^{n-2}(x), ..., x, y) \wedge \Delta^{\gamma}[\overline{s}_{\sigma'}](x) \wedge \xi_1(\overline{x}),$$

where $u \in T_n^{\beta}$ is a branch of length $\gamma \leq \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta$; where ξ_1 is a pseudo-quantifier-free formula which does not involve y; where $\sigma'(1) < ... < \sigma'(j) < n$; and where $s_{\sigma'(i)}$ is a branch in $T^{\beta}_{\sigma'(i)}$ for $i \leq j$.

Now consider whether the branch $u \in T_n^{\beta}$ is isolated. If it is isolated by some level $\varepsilon < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta$ then we have

$$\mathcal{T}_{\beta+1} \vdash \forall \overline{x}, y \ (\theta_{n,u}(f^{n-2}(x), ..., x, y) \leftrightarrow \theta_{n,u\restriction\varepsilon}(f^{n-2}(x), ..., x, y)).$$

But then by induction there is a pseudo-quantifier-free $\xi_2(x)$ for which

$$\mathcal{T}_{\beta} \vdash \forall \overline{x} \ (\exists y \ \theta_{n,u\restriction\varepsilon}(f^{n-2}(x), ..., x, y) \wedge \Delta^{\varepsilon}[\overline{s}_{\sigma'}](x)) \leftrightarrow \xi_2(x)).$$

Now let $\xi_{\varphi}(\overline{x}) := \xi_2(x) \wedge \xi_1(\overline{x})$, so that

$$\mathcal{T}_{\beta+1} \vdash \forall \overline{x}((\exists y \varphi(\overline{x}, y)) \leftrightarrow \xi_{\varphi}(\overline{x})).$$

If u is not isolated, then we will define $\xi_3(x)$ so that we may take $\xi_{\varphi}(\overline{x})$ to be the formula

$$\xi_3(x) \wedge \Delta^{\gamma}[\overline{s}_{\sigma'}](x) \wedge \xi_1(\overline{x}).$$

There are three possibilities for the non-isolated branch u. If $u \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta = b^{\beta}_{n,k}$ for some $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, let $\xi_3(x) := U_{n-1,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta+k\cdot 2}(x)$. If $u \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta = q^{\beta}_{n,t^{\delta}(k)}$ for some $k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, let $\xi_3(x) := U_{n-1,\mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta+k\cdot 2+1}(x)$. If $u \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)\beta = Q^{\beta}_n$, let $\xi_3(x)$ be empty. In all three possibilities, the (EC) axioms (3) and (4) tell us that

$$\mathcal{T}_{\beta+1} \vdash \forall x((\exists y\ \theta_{n,u}(f^{n-2}(x), ..., x, y) \leftrightarrow \xi_3(x)).$$

Hence

$$\mathcal{T}_{\beta+1} \vdash \forall \overline{x}((\exists y \varphi(\overline{x}, y)) \leftrightarrow \xi_\varphi(\overline{x})),$$

establishing the lemma in all cases. ⊣

Theorem 2.8. *Let $\delta < \alpha$. Suppose $\{T_n^\delta : n < \omega\}$ is a set of trees which satisfies* TP(δ)*. Then for $\beta \leq \delta$, the corresponding theories $\mathcal{T}_\beta$ defined in terms of such trees are complete in $\mathcal{L}_\beta$. Hence $\mathcal{T}_\delta$ is a complete theory of rank δ.*

Proof. By Lemma 2.6, $\mathcal{T}_\beta$ eliminates quantifiers down to pseudo-quantifier-free rank β formulas. But the truth of pseudo-quantifier-free sentences of rank β is determined by $\mathcal{T}_\beta$, and so $\mathcal{T}_\beta$ is complete in the language $\mathcal{L}_\beta$. Therefore $\mathcal{T}_\delta$ is complete in $\mathcal{L}_\delta$.

It has rank δ as each theory $\mathcal{T}_\beta$ has types that are non-principal in $\mathcal{L}_\beta$ but implied by atoms of $\mathcal{T}_\delta$ (when $\beta < \delta$), and so there are atoms of arbitrarily high rank below δ. ⊣

Theorem 2.9. *The theory $\mathcal{T}_\alpha$ determined by trees $\{T_n^\alpha : n < \omega\}$ satisfying* TP(α) *has $|\alpha|$ many non-principal types, and the complexity of the types is at least the least complexity of the non-isolated branches of trees in $\{T_n^\alpha : n < \omega\}$.*

Proof. Given a formula, we may reduce it to a pseudo-quantifier-free formula using Lemma 2.6. A complete n-type of T_α can therefore be written in the form

$$\Delta^0_{G,S}(\overline{x}_n) \wedge \bigwedge_{b \text{ a source vertex of } G} \Delta^\alpha[\overline{s}(b)_{n_b}](x_b),$$

where each $s(b)_i$ is a branch through the tree T_i^α and each $\overline{s}(b)_{n_b} := s(b)_1, ..., s(b)_{n_b}$.

Now note that the complexity of a type is the same as the complexity of the set of branches

$$\{s(b)_i : b \text{ a source vertex of } G, \text{ and } i \leq n_b\}.$$

Furthermore, a type is non-principal precisely when one of these branches is non-isolated. So in particular, the complexity of a non-principal type is at least that of the least complex non-isolated branch.

TP(δ) ensures that there are exactly $|\delta|$ many branches in each tree T_n^δ, and so TP(α) guarantees that there are $|\alpha|$ many non-principal types. $\dashv$

3 Priority

3.1 Requirements and Witnesses

We will build trees $\{T_n : n < \omega\}$ satisfying TP(α) via a Δ_1^α construction. The non-isolated branches are α-recursively partitioned into the three collections of branches: the regenerating branches, priority branches, and seed branches.

Let B be the set of symbols denoting all such *named branches*:

$$B = \{b_{n,k} : n < \omega,\ k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\} \cup \{q_{n,\delta} : n < \omega,\ \delta < \alpha\} \cup \{Q_n : n < \omega\}.$$

Definition 3.1. *A* work stage σ *is a successor of a successor ordinal with* $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\ +\ 1\ <\ \sigma\ < \alpha$.

For the first $\mathrm{t}\sigma 2\mathrm{p}(\alpha) + 1$ many stages of the construction we will ignore the priority and merely build trees. Then, once the priority argument starts at stage $\mathrm{t}\sigma 2\mathrm{p}(\alpha) + 2$ we may potentially address any requirement.

For each named branch $s \in B$ and for all $\xi < t^\sigma(\xi)$, at each work stage σ, we will build approximations s^σ (the primary candidate at stage σ) which are non-isolated at level σ and s^σ_ξ (the $t^\sigma(\xi)^{\mathrm{th}}$ potential candidate at stage σ). These non-isolated primary candidates will be approximated in a tame Σ_2 manner by their isolated approximations at stages $\sigma < \alpha$, and are named by the symbols:

$$B^\sigma = \{b^\sigma_{n,k} : n < \omega, k < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\} \cup \{q^\sigma_{n,\delta} : n < \omega, \delta < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\sigma\} \cup \{Q^\sigma_n : n < \omega\}.$$

Additionally, we will build witnesses $Ws^\sigma(\xi) = (w, v) \in \alpha \times 2$ such that the w^{th} bit of the primary candidate for the requirement $Rs(\xi)$ at stage σ is v. The goal of requirement $Rs(\xi)$ is to diagonalize against the $t(\xi)^{th}$ partial α-recursive function on branch s.

Fix $s \in B$. We will see that the witness approximations $Ws^\delta(\xi)$ for fixed δ, ξ are Δ^α_1. But the limiting value $Ws(\xi)$ is a tame Σ^α_2 function of ξ; it is reset periodically, though it eventually settles on arbitrarily large initial segments. When a higher priority requirement switches from its primary candidate to a potential candidate, it resets lower priority witnesses. Also, when our approximation to $t(\xi)$ changes, we discard its witnesses and those of lower priority. Similarly, we will see that the Δ^α_1 values s^δ tend to the limiting value of s in a tame Σ^α_2 manner.

Requirements: For $s \in B$ and $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, we say that $Rs(\xi)$ is *satisfied* iff $s \neq \chi_{V_{t(\xi)}}$. At any stage $\sigma < \alpha$, each of $Rs^\sigma(\xi)$ is said to be in exactly one of the states *unhappy, addressed,* or *α-finitely satisfied,* according to the construction. If $Rs^\sigma(\xi)$ is either *addressed* or *α-finitely satisfied,* we call $Rs^\sigma(\xi)$ *happy.* We say that $Rs(\xi)$ is in a particular state at stage σ iff $Rs^\sigma(\xi)$ is in that state.

At each stage $\sigma < \alpha$, we give a Δ^α_1 construction of trees $\{T^\sigma_n : n \in \omega\}$ satisfying $\mathrm{TP}(\sigma)$. In the process, we define the branches in B^σ, which constitute the non-isolated branches of $\{T^\sigma_n : n \in \omega\}$. We also define witnesses $Ws^\sigma(\xi)$ for $s^\sigma \in B^\sigma$.

We inductively verify the following property $*\mathrm{TP}(\sigma)$, for each $\sigma < \alpha$:

$*$**TP**(σ) : The trees $\{T^\sigma_n : n \in \omega\}$ satisfy $\mathrm{TP}(\sigma)$. Each non-isolated branch $s \in B^\sigma$ has σ many potential candidates, and if $t^\sigma(\xi) < t^\sigma(\zeta)$ then $(Ws^\sigma_\xi)_0 < (Ws^\sigma_\zeta)_0$.

Proposition 3.2. *At the end of a stage $\sigma > \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, the trees $\{T^\sigma_n : n \in \omega\}$ satisfy $*\mathrm{TP}(\sigma)$.*

Following the construction, we will prove Proposition 3.2 by induction, and hence, during the construction, at work stages $\delta+2$ we may assume that $*\mathrm{TP}(\delta+1)$ holds.

3.2 Construction

Stage 1**.** For each $n < \omega$, set $T^1_n := \{s \in 2^{<\mathrm{t}\sigma 2\mathrm{p}(\alpha)} : s$ has at most two 0's$\}$. Then the isolated branches are $\{s^\frown 1^\omega : s \in 2^{<\mathrm{t}\sigma 2\mathrm{p}(\alpha)}$ has exactly two 0's$\}$. Among the non-isolated branches, we designate the regenerating branches $b^1_{n,k} := 1^k 0 1^\omega$ and the seed branch $Q^1_n := 1^{\mathrm{t}\sigma 2\mathrm{p}(\alpha)}$. There are not yet any priority branches. All witnesses Ws^1_ξ are undefined and all requirements $Rs^1(\xi)$ are unhappy, in which state they remain until the first work stage.

Stage γ (where γ is a successor such that $1 < \gamma < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ or $\gamma = \omega\delta + 1$ for $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \leq \delta < \alpha$). Here we build trees $\{T^\gamma_n : n < \omega\}$ satisfying $\mathrm{TP}(\gamma)$, and ignore the priority argument. First we compute $t^\gamma \restriction (\gamma + 1)$. For every terminal branch $u \in T^{\gamma-1}_n$, extend u to a string v of length $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma$ using 0 as necessary to ensure $\mathrm{NegB}^{\gamma-1}(v) \subseteq \mathrm{NegB}^\gamma(v)$ and $\mathrm{tNegQ}^{\gamma-1}_{\gamma-1}(v) \subseteq \mathrm{tNegQ}^\gamma_{\gamma-1}(v)$, but extending by 1 elsewhere. Add all such v to T^γ_n. If u is a non-isolated branch of $T^{\gamma-1}_n$, hence named by some symbol $s^{\gamma-1} \in B^{\gamma-1}$ (as justified afterwards in Lemma 3.4), denote the new string v by the corresponding symbol $s^\gamma \in B^\gamma$.

Let s^γ be such a newly-named branch. We ensure that s^γ is non-isolated by adding the following approximations to T^γ_n. For each $k < \omega$ add to T^γ_n the branch defined by

$$u_k(\beta) := \begin{cases} 1 - s^\gamma(\beta) & : \quad \beta = \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + 2k, \\ s^\gamma(\beta) & : \quad \text{otherwise.} \end{cases}$$

Further add the named branch

$$q^\gamma_{n,\gamma}(\beta) := \begin{cases} 0 & : \quad \beta = \mathrm{t}\sigma 2\mathrm{p}(\alpha)(\gamma - 1) + 2k, \\ Q^\gamma_n(\beta) & : \quad \text{otherwise,} \end{cases}$$

and for $1 < k < \omega$ add to T_n^γ the approximations defined by

$$u'_k(\beta) := \begin{cases} 1 - q_{n,\gamma}^\gamma(\beta) & : \quad \beta = \mathrm{t}\sigma 2\mathrm{p}(\alpha)\gamma + 2k, \\ q_{n,\gamma}^\gamma(\beta) & : \quad \text{otherwise.} \end{cases}$$

Stage $\delta + 2$, for $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \leq \delta < \alpha$, i.e., a work stage. In Step 0, we will first note where our approximation to t has changed, and discard our past work which used the old approximation. We then, for each $s \in B^{\delta+2}$, attempt to make more requirements happy, by α-finitely satisfying some previously addressed requirement whose witness has just been enumerated into the relevant α-recursively enumerable set (in Step 1), and by making some previously unhappy requirement happy (in Step 2). Finally, in Step 3 we build the next level of trees, incorporating these changed strings but still satisfying $\mathrm{TP}(\delta + 2)$.

Step 0. Compute $t^{\delta+2} \upharpoonright (\delta + 3)$. Let ψ be the least value on which $t^{\delta+2}(\psi) \neq t^{\delta+1}(\psi)$. For all $\xi \geq \psi$, reset $Rs^{\delta+2}(\xi)$ to unhappy. (We do not discard the witnesses $Ws_\xi^{\delta+1}$ for potential candidates, as our approximation to t may jump forward again at some later stage.)

For each $s^{\delta+1} \in B^{\delta+1}$ we perform the following two steps.

Step 1. Let $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ be the least ordinal for which $Rs^{\delta+1}(\xi)$ is addressed but $(Ws_\xi^{\delta+1})_0 \in V_{t^{\delta+2}(\xi)} \setminus V_{t^{\delta+1}(\xi)}$, if such ξ exist. If there is no such ξ, set $s^{\delta+2} \upharpoonright \mathrm{t}\sigma 2\mathrm{p}(\alpha)(\delta + 1) := s^{\delta+1}$ and proceed to Step 2 for s. Otherwise, $t^{\delta+2}(\xi) \downarrow$ implies that $t^{\delta+2}(\xi) < \delta + 2$ and so the branch $s^{\delta+1}$ has a potential candidate $s^{\delta+1}_{t^{\delta+2}(\xi)}$, as $t^{\delta+2}(\xi) = t^{\delta+1}(\xi)$, and $*\mathrm{TP}(\delta{+}1)$ holds by hypothesis. Then set $s^{\delta+2} \upharpoonright \mathrm{t}\sigma 2\mathrm{p}(\alpha)(\delta+$

$1) := s^{\delta+1}_{t^{\delta+1}(\xi)}$, thereby injuring $Rs(\zeta)$ for all $\zeta > \xi$. Reset all $Rs^{\delta+2}(\zeta)$ to unhappy, for $\zeta > \xi$. Maintain the witness location $(Ws^{\delta+2}_\xi)_0 := (Ws^{\delta+1}_\xi)_0$ but change the value $(Ws^{\delta+2}_\xi)_1 := 0$; now $Rs^{\delta+2}(\xi)$ is α-finitely satisfied.

Step 2. If all requirements $Rs^{\delta+1}(\xi)$ for $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ are happy, do nothing. Also, if $\delta+1$ is not in the range of $t^{\delta+2}$, do nothing. Otherwise, let ξ be least for which $Rs^{\delta+1}(\xi)$ is unhappy and for which $(\exists k < \mathrm{t}\sigma 2\mathrm{p}(\alpha))(t^{\delta+2}(k) \downarrow = \delta+1)$. Let k be the least such ordinal. Let n be the first subscript of the symbol in $B^{\delta+1}$ denoted by $s^{\delta+1}$ (or equivalently, the subscript of the tree $T^{\delta+1}_n$ it is in). We now define the witness $Ws^{\delta+2}_\xi$. Note that $\delta+1 \notin \mathrm{tNegQ}^\delta_\delta(s^{\delta+1})$ and so, as we extend $s^{\delta+1}$ the next $\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ many steps, either choice at height $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\delta + k\cdot 2+1$ is permitted. Thus we may choose according to the priority branch $q^{\delta+1}_{n+1,\delta+1}$. Let $(Ws^{\delta+2}_\xi)_0 = \mathrm{t}\sigma 2\mathrm{p}(\alpha)\delta + k\cdot 2+1$. If $(Ws^{\delta+2}_\xi)_0 \in V^{\delta+2}_{t^{\delta+2}(\xi)}$ the let $(Ws^{\delta+2}_\xi)_1 := 0$ and set $Rs^{\delta+2}(\xi)$ to α-finitely satisfied. Otherwise, let $(Ws^{\delta+2}_\xi)_1 := 1$ and set $Rs^{\delta+2}(\xi)$ to addressed.

Step 3. We now construct trees $\{T^{\delta+2}_n : n < \omega\}$ satisfying $\mathrm{TP}(\delta+2)$, but respecting the choices of the priority argument. We proceed exactly as in the previous case (using $\gamma := \delta+2$) except that whenever we had defined part of a non-isolated branch $s^{\delta+2}$ in Step 1, we extend $s^{\delta+2} \restriction \mathrm{t}\sigma 2\mathrm{p}(\alpha)(\delta+1)$ (as defined there), instead of the branch $s^{\delta+1}$. This accounts for the injurious change of Step 1, while still allowing pseudo-predicates to be constructed from the original $s^{\delta+1}$. The requirements which have recently been made happy are automatically incorporated into the trees via our restrictions on $\mathrm{NegB}^{\delta+2}$ and $\mathrm{tNegQ}^{\delta+2}_{\delta+1}$.

Stage $\omega\delta$ (for some $\delta < \alpha$). We simply take the unions of the trees:

$$T_n^{\omega\delta} := \bigcup_{\gamma<\omega\delta} T_n^{\gamma}.$$

This is justified as $\mathrm{TP}(\gamma)$ holds for $\gamma < \omega\delta$.

3.3 Verification

Lemma 3.3. *For each $\sigma < \alpha$, the construction of $\{T_n^\sigma : n < \omega\}$ is Δ_1^α.*

Proof. Our approximation t^σ was chosen to be Δ_1^α. By induction, assume $\{T_n^\delta : n < \omega, \delta < \sigma\}$ to be Δ_1^α. Each of the primary candidates s^σ and potential candidates s_ξ^σ were defined α-recursively in terms of the previously constructed trees, as were the witnesses Ws_ξ^σ and the sets $\mathrm{NegB}^\sigma(u)$ and $\mathrm{tNegQ}_\tau^\sigma(u)$, for $u \in T_n^\sigma$, $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, and $\tau < \sigma$. But then the new branches in $\{T_n^\sigma : n < \omega\}$ are Δ_1^α via their α-recursive definitions (in various cases) in terms of the above data. $\dashv$

Lemma 3.4. *Requirements never switch from α-finitely satisfied to addressed. For $\sigma < \alpha$, each non-isolated branch u of T_n^σ is named by some $s^\sigma \in B^\sigma$, and those named by distinct symbols of B^σ are distinct branches. Similarly, each non-isolated branch u of T_n^α is named by some $s \in B$, and those named by distinct symbols of B are distinct branches.*

Proof. Note that in Step 1, requirements either become unhappy or go from unhappy to α-finitely satisfied, and in Step 2 go from unhappy to happy. Nowhere else in the construction do requirements change state.

If $u \in T_n^\sigma$ is a non-isolated branch, by the construction it either extends a non-isolated branch (hence named by induction) and remains so named, or is created in Stage σ and assigned a new name. If $u \in T_n^\alpha$ is non-isolated, then so is some initial segment, which we have just shown is named. By the construction, this name persists.

The construction gives distinct names to new named branches and preserves as initial segments the old ones, so that all distinctly named branches differ at each stage σ, and in the limit in T^α_n. $\dashv$

Injuries never occur at limit stages, and so we may define the injury and action sets $Is(\xi)$ as follows.

Definition 3.5. *For $s \in B$ and $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, set*

$$\begin{aligned} Is(\xi) &:= \{\sigma : Rs^{\sigma-1}(\xi) \textit{ is happy and } Rs^{\sigma}(\xi) \textit{ is unhappy}\}, \textit{and} \\ As(\xi) &:= \{\sigma : Rs^{\sigma-1}(\xi) \textit{ is unhappy and } Rs^{\sigma}(\xi) \textit{ is happy}\}. \end{aligned}$$

If $\mathrm{t}\sigma 2\mathrm{p}(\alpha) > \mathrm{gc}(\alpha)$ then we further define, for $\gamma < \sigma 2\mathrm{cf}(\alpha)$,

$$Js(\gamma) := \bigcup \{Is(\xi) : \mathrm{gc}(\alpha) \cdot \gamma \leq \xi \leq \mathrm{gc}(\alpha) \cdot (\gamma + 1)\}.$$

Note that $As(\xi)$ does not count transitions from *addressed* to *α-finitely satisfied.*

Lemma 3.6. *Let $s \in B$, $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, and $\gamma < \sigma 2\mathrm{cf}(\alpha)$. Let $\mu > \xi$ and $\mu' > \gamma$ be infinite α-cardinals. Then $Is(\xi)$ is α-finite and of α-cardinality $< \mu$. If $\mathrm{t}\sigma 2\mathrm{p}(\alpha) > \mathrm{gc}(\alpha)$ then $Js(\gamma)$ is α-finite and of α-cardinality $< \mu'$.*

Proof. Fix $s \in B$ and $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$, and take $\sigma < \alpha$ such that the α-recursive approximation t^τ is correct up to ξ for all stages from σ on, i.e., $(\forall \tau \geq \sigma)\ t^\tau \restriction (\xi + 1) = t \restriction (\xi + 1)$. (This is possible by the tame Σ^α_2 definition of t and its approximations.) By Lemma 1.21, we have two cases, depending on which of $\mathrm{t}\sigma 2\mathrm{p}(\alpha)$ and $\mathrm{gc}(\alpha)$ is larger.

Case 1: $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \leq \mathrm{gc}(\alpha)$.

Assume by induction that for all $\delta < \xi$, $Is(\delta)$ is α-finite and that there is a regular α-cardinal $\zeta > \xi$ for which each $Is(\delta)$ has α-cardinality $< \zeta$.

(If $\mathrm{gc}(\alpha)$ is an α-cardinal $< \alpha$, then if $\mathrm{gc}(\alpha)$ is α-regular we may take $\zeta = \mathrm{gc}(\alpha)$, and if $\mathrm{gc}(\alpha)$ is α-singular then there is a regular α-cardinal $\zeta > \xi$. If $\mathrm{gc}(\alpha) = \alpha$ then there is also a regular α-cardinal $\zeta > \xi$. By hypothesis, for $\delta < \xi$ the set $Is(\delta)$ has α-cardinality $< \zeta$.)

By the hypothesis on Case 1, $\xi < \mathrm{gc}(\alpha)$. Consider the sets $Is(\delta)$. Considered as a function of τ, the state of the requirement $Rs^{\tau}(\xi)$ is Δ_1^{α}, because the entire construction of $\{T_n^{\tau} : n < \omega\}$ (including assignment of requirement states) is Δ_1^{α}. Hence $\{Is(\delta) : \delta < \xi\}$ is simultaneously α-recursively enumerable, by simulating the construction of $\{T_n^{\tau} : n < \omega\}$ and noting at which stages the relevant injuries occur. Further, they are all of α-cardinality less than ζ. So, by Lemma 1.12, $\bigcup\{Is(\delta) : \delta < \xi\}$ is α-finite and of α-cardinality less than ζ.

Note that $Is(\delta)$ and $As(\delta)$ are interlaced, i.e., between any two elements of one is an element of the other. So $As(\delta)$ is α-finite iff $Is(\delta)$ is, and their α-cardinalities differ by 0 or 1. Hence $\bigcup\{As(\delta) : \delta < \xi\}$ is also α-finite and of α-cardinality less than ζ.

But note that $Rs(\delta)$ is only injured by higher priority requirements, and so $Is(\delta) \subseteq \bigcup\{As(\delta) : \delta < \xi\}$. So $Is(\xi)$ is a subset of some ordinal less than $\mathrm{gc}(\alpha)$. By Corollary 1.14, $\mathrm{gc}(\alpha) \leq \alpha^*$. Hence by Lemma 1.7, $Is(\xi)$ is α-finite (and of α-cardinality $< \zeta$). Since we could have chosen ζ to be less than or equal to any chosen α-cardinal $\mu > \xi$ (either μ is α-regular, in which case take $\zeta = \mu$, or μ is α-singular, in which case take $\xi < \zeta < \mu$), we have the desired bound on the α-cardinality of $Is(\xi)$.

Case 2: $\mathrm{t}\sigma 2\mathrm{p}(\alpha) = \mathrm{gc}(\alpha) \cdot \sigma 2\mathrm{cf}(\alpha)$.

Consider the block of length $\mathrm{gc}(\alpha)$ in which ξ lies. Let γ be the unique ordinal such that $\mathrm{gc}(\alpha) \cdot \gamma \leq \xi < \mathrm{gc}(\alpha) \cdot (\gamma + 1)$. We induct on ξ to simultaneously prove the two claims. Assume by induction that there is some regular α-cardinal $\zeta > \xi$ such that for all $\delta < \xi$, $Is(\delta)$ is α-finite and of α-cardinality less than ζ, and that there is some regular α-cardinal $\zeta' > \gamma$ such that for all $\varepsilon < \gamma$, $Js(\varepsilon)$ is α-finite and of α-cardinality less than ζ'. (As in Case 1, we may choose ζ and ζ' to be α-regular and less than or equal to any chosen α-cardinals $\mu > \xi$ and $\mu' > \gamma$, respectively.)

$Js(\varepsilon)$ is a Σ_2^α function (as $Is(\delta)$ is simultaneously α-recursively enumerable as in Case 1). So $\bigcup\{Js(\varepsilon) : \varepsilon < \gamma\}$ is α-finite, as $\gamma < \sigma 2\mathrm{cf}(\alpha)$. Hence we may pick $\sigma' = \sup(\sigma, \sup(\bigcup\{Js(\varepsilon) : \varepsilon < \gamma\}))$ such that by stage σ' the requirements in block ε have settled for all $\varepsilon < \gamma$.

Set $Is'(\delta) = Is(\delta) \setminus \sigma'$. Now proceed as in Case 1 to show that for $\mathrm{gc}(\alpha) \cdot \gamma \leq \delta < \xi$, the set $Is'(\delta)$ is α-finite and of α-cardinality less than ζ, and so $Is'(\xi)$ is α-finite and of α-cardinality less than ζ. Hence $Is(\xi) \subseteq Is'(\xi) \cup \sigma'$ is α-finite (and of α-cardinality less than ζ).

To show that $Js(\gamma)$ is α-finite, it suffices to consider only activity after stage σ' (as the injuries before this stage are bounded by σ'). Denote by U the set $\bigcup\{Is'(\delta) : \mathrm{gc}(\alpha) \cdot \gamma \leq \delta < \mathrm{gc}(\alpha) \cdot (\gamma + 1)\}$. Now we show that U is α-finite and of α-cardinality less than ζ'.

First we show that each $Is'(\delta)$ is α-finite and of α-cardinality less than ζ, for $\mathrm{gc}(\alpha) \cdot \gamma \leq \delta < \mathrm{gc}(\alpha) \cdot (\gamma + 1)$. We proceed by an induction on δ as in Case 1. Assume that, for $\mathrm{gc}(\alpha) \cdot \gamma \leq \eta < \delta$, the set $Is'(\eta)$ is α-finite and of α-cardinality less than ζ. Again we obtain, by Lemma 1.12, that $\bigcup\{Is'(\eta) : \mathrm{gc}(\alpha) \cdot \gamma < \eta < \delta\}$ is α-finite and of α-cardinality less than ζ (as the union is over fewer than ζ many terms). An identical argument about interlacing the sets $As'(\eta)$ (analogously defined) shows that $Is'(\delta)$ is α-finite and of α-cardinality less than ζ, as desired. This also implies that U is of α-cardinality less than ζ'.

Now we show that the union U of these sets $Is'(\delta)$ is α-finite. As before, our definition of $Is'(\delta)$ implies that there is a simultaneous α-recursive enumeration given by some $\iota : \mathrm{gc}(\alpha) \times \alpha \to U$, a total α-recursive function which is surjective but not necessarily injective. Note also that U is partitioned by the sets $Is'(\delta)$; the ranges of $\iota(\delta, -)$ are disjoint for distinct δ. Consider the injective partial function $\rho : \alpha \to \mathrm{gc}(\alpha)$, which sends $\tau \in U$ to β where τ is the β-th element enumerated into the range of $\iota(\delta, -)$. Now ρ is not necessarily partial α-recursive, because we can't α-recursively enumerate $\alpha \setminus U$. However, define $\nu : \alpha \to \mathrm{gc}(\alpha) \cdot \mathrm{gc}(\alpha)$ by $\nu(\tau) = (\rho(\tau), \delta - \mathrm{gc}(\alpha) \cdot \gamma)$ where $\tau \in Is'(\delta)$. Note that ν is injective, and *is* partial α-recursive: to see if $\nu(\tau) = (\vartheta, \psi)$, one α-recursively computes ι on inputs (x, y) with $x \leq \psi$ and $y \leq \vartheta$.

Suppose U is not α-finite. Then let $\rho : \alpha \to U$ be an injective α-recursive map witnessing such. We may compose ρ with ν to obtain an injective α-recursive map from α to $\mathrm{gc}(\alpha) \cdot \mathrm{gc}(\alpha)$. But then $\mathrm{gc}(\alpha)$ is not α-finite either, i.e., $\alpha^* \leq \mathrm{gc}(\alpha)$. By Corollary 1.19, $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \leq \alpha^*$, and so we have $\mathrm{t}\sigma 2\mathrm{p}(\alpha) \leq \mathrm{gc}(\alpha)$, contradicting our Case 2 hypothesis. $\dashv$

Lemma 3.7. *Let $s \in B$ and $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$. The witnesses and named branches settle on arbitrarily large initial segments, i.e. there is tame Σ_2^α function of $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$ defining Ws_ξ and of $\delta < \alpha$ defining $s \restriction \delta$. Further, either*

(a) $(\exists \sigma < \alpha)(\forall \tau > \sigma) Rs^\tau(\xi)$ *is addressed, or*

(b) $(\exists \sigma < \alpha)(\forall \tau > \sigma) Rs^\tau(\xi)$ *is α-finitely satisfied.*

Proof. Fix ξ, δ, and s. We show that there is a stage beyond which Ws_ζ and s^δ are settled, and $Rs(\zeta)$ is happily settled, for all $\zeta \leq \xi$. By Lemma 3.6, we may pick some stage σ beyond which $Rs(\zeta)$ is not injured for $\zeta \leq \xi$, and for which $\mathrm{t}\sigma 2\mathrm{p}(\alpha)\sigma > \delta$. In particular, $t^\sigma \restriction (\xi + 1)$ is correct.

By induction we may further assume that (for $\zeta < \xi$) each $Rs(\zeta)$ has settled (by stage σ) to particular happy state. If $Rs^\sigma(\xi)$ is already α-finitely satisfied, then it remains so forever, and Ws_ξ and $s^\sigma \restriction \delta$ are also already correct, as nothing in the construction will cause them to change, and since $\delta < \mathrm{t}\sigma 2\mathrm{p}(\alpha)\sigma$.

If $Rs^\sigma(\xi)$ is addressed, and happens not to change later, then again Ws_ξ and $s^\sigma \restriction \delta$ are correct. Suppose $Rs^\sigma(\xi)$ is addressed, but it later changes. It cannot become unhappy, as all higher-priority requirements have settled. Thus it must be later α-finitely satisfied in Step 1 of some stage σ_0. As it is never again injured, it remains in this state forever, and $Rs^{\sigma_0}(\xi)$, $Ws_\xi^{\sigma_0}$, and $s^{\sigma_0} \restriction \delta$ are the final values.

Finally, suppose that $Rs^\sigma(\xi)$ is unhappy. The construction acts on $Rs(\xi)$ in Step 2 of the least work stage $\sigma_0 > \sigma$, because all higher-priority requirements are happy. If it is possible to α-finitely satisfy it, the construction does so, in which case $Rs^{\sigma_0}(\xi)$, $Ws_\xi^{\sigma_0}$, and $s^{\sigma_0} \restriction \delta$ are the final values, as above. If not, it is addressed in stage σ_0. As before, if it remains addressed forever, these are also the final values, and if it changes (once more, to α-finitely satisfied) then the witnesses and branch stabilize by this later stage. $\dashv$

Lemma 3.8. *Let $s \in B$ and $\xi < \mathrm{t}\sigma 2\mathrm{p}(\alpha)$.*

(a) Suppose there is a stage $\sigma < \alpha$ such that $(\forall \tau > \sigma) Rs^\tau(\xi)$ is addressed. Then $Rs(\xi)$ is satisfied.

(b) Suppose there is a stage $\sigma < \alpha$ such that $(\forall \tau > \sigma) Rs^\tau(\xi)$ is α-finitely satisfied. Then $Rs(\xi)$ is satisfied.

Proof. In either case, let σ_0 be the least such stage. Let $\sigma \geq \sigma_0$ be the least stage beyond which $t^\sigma \upharpoonright (\xi+1)$ has settled.

(a) We have that $(Ws^\sigma_\xi)_0 \notin V^\sigma_{t^\sigma_\xi}$ while $(Ws^\sigma_\xi)_1 = 1$. Also, at no later stage τ does $(Ws^\sigma_\xi)_0$ enter $V^\tau_{t(\xi)}$, or else we later act to α-finitely satisfy $Rs(\xi)$. Hence $s \neq \chi_{V_{t(\xi)}}$, as they differ on input $(Ws_\xi)_0$, and so $Rs(\xi)$ is satisfied.

(b) By the construction, Ws_ξ has already settled, i.e., $(\forall \tau \geq \sigma) Ws_\xi = Ws^\tau_\xi$, since if it changed there would be an injury to $Rs(\xi)$ past stage σ. Our Σ^α_1 enumeration of $V_{t(\xi)}$ is from below, and $t(\xi)$ has settled by σ; hence $\chi_{V_{t(\xi)}}((Ws_\xi)_0) = \chi_{V^\sigma_{t(\xi)}}((Ws^\sigma_\xi)_0) = 1$. Similarly, $(Ws_\xi)_1$ remains 0, and so $Rs(\xi)$ is satisfied. ⊣

We now prove Proposition 3.2, namely that $*\mathrm{TP}(\sigma)$ holds at stages $\sigma >$ tσ2p(α).

Proof of Proposition 3.2. Let $\sigma = \delta + 2$ be a work stage. The trees cohere by virtue of Step 3. The named branches are continuously defined by Lemma 3.7. Steps 1 and 2 are careful to preserve our restrictions on NegB and tNegQ. The non-isolated and potentially non-isolated branches take value 1 at even heights (except for the restriction on priority types) because of our choices in Steps 2 and 3, which also guarantees infinitely many branches that take value 1 where required.

At $\sigma =$ tσ2p(α) and successors of limits $\sigma >$ tσ2p(α), the trees were extended precisely so as to preserve restrictions on NegB and tNegQ while constructing new potentially non-isolated branches approximating the non-isolated branches as required by TP(σ).

For limit stages $\sigma >$ tσ2p(α), the property TP(σ) holds automatically by induction. Thus TP(σ) is satisfied.

Furthermore, we created (or renamed) countably many new potential candidates at each stage for which our approximation to t moved forward, and there are tσ2p(α) many such stages. Sometimes their witnesses were temporarily abandoned, but upon becoming active again, the order of the witness nodes was again made monotone in our approximation to t, and so $*$TP(σ) holds. $\dashv$

Theorem 3.9. *There is a Δ_1^α set of trees $\{T_n^\alpha : n < \omega\}$ satisfying* TP(α) *with no non-isolated branches α-recursively enumerable.*

Proof. By Lemma 3.3, for each stage σ we have that $\{T_n^\sigma : n < \omega\}$ is Δ_1^α. By Proposition 3.2, if σ is a work stage, then $\{T_n^\sigma : n < \omega\}$ satisfies TP(σ). The property TP(α) holds, as it is just $\bigcup_{\sigma<\alpha}$ TP(σ), and work stages are cofinal in α. By TP(α), the trees cohere and are continuously defined, and so $\{T_n^\alpha : n < \omega\}$ is Δ_1^α.

Let $s \in B$ and $\xi <$ tσ2p(α). By Lemmas 3.7 and 3.8, each $Rs(\xi)$ is satisfied. Hence $s \neq \chi_{V_{t(\xi)}}$. By Lemma 3.4, each non-isolated branch of a tree in $\bigcup_{\sigma<\alpha}$ TP(σ) is named by an element of B. The function t is surjective onto α, so every α-recursively enumerable set is equal to $V_{t(\xi)}$ for some $\xi <$ tσ2p(α). Hence no non-isolated branch is α-recursively enumerable. $\dashv$

4 Model (for $\omega_1^{\mathrm{CK}} \leq \alpha < \omega_1$)

Throughout this section, let $\omega_1^{\mathrm{CK}} \leq \alpha < \omega_1$. Fix $\{T_n^\alpha : n < \omega\}$ as in Theorem 3.9. Then, by Theorem 2.9, the corresponding theory $\mathcal{T}_\alpha$ has some, but only countably many, non-principal types, none of which are Σ_1^α. By construction, $\mathcal{T}_\alpha$ is a complete and consistent Scott theory in the language $\mathcal{L}_{\alpha,\omega}$, as shown in Theorems 2.3 and 2.8. Thus we obtain

Corollary 4.1. *There is a complete and consistent Δ_1^α Scott theory $\mathcal{T}_\alpha$ in the language $\mathcal{L}_{\alpha,\omega}$ with some, but only countably many, non-principal types, none of which are Σ_1^α.*

We now construct a countable structure $\mathcal{A}$ with $\mathcal{T}_\alpha$ as its Scott theory, which omits the non-principal types, and which preserves the Σ_1 admissibility of α. We mostly follow Millar-Sacks [12]. A similar method is suggested by Grilliot [6].

Theorem 4.2. *There is a countable structure $\mathcal{A}$ for which*

(1) $\omega_1^{\mathcal{A}} = \alpha$;

(2) the $\mathcal{L}_{\alpha,\omega}$-theory of $\mathcal{A}$ is Δ_1^α;

(3) the Scott rank of $\mathcal{A}$ is α;

(4) $\mathcal{A}$ is an atomic model of its $\mathcal{L}_{\alpha,\omega}$-theory;

(5) the $\mathcal{L}_{\alpha,\omega}$-theory of $\mathcal{A}$ is not $\aleph_0$-categorical; and

(6) no non-principal type of the $\mathcal{L}_{\alpha,\omega}$-theory of A is Σ_1^α.

Proof. As in Section 1, let $\mathcal{T}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ be the complete theory of $\mathcal{A}$ in $\mathcal{L}^{\mathcal{A}}_{\alpha,\omega}$. We will use Barwise compactness (modified to effectively omit certain types) to construct a countable structure $\mathcal{A}$ for which $\mathcal{T}^{\mathcal{A}}_{\omega_1^{\mathcal{A}},\omega}$ is the theory $\mathcal{T}_\alpha$ from Corollary 4.1.

We first construct a Σ_1 admissible end extension $\mathcal{B}$ of $L(\alpha)$ with a constant symbol whose realization is the desired model $\mathcal{A}$.

Let (F) be the following set of sentences:

(F1) The atomic diagram within $\mathcal{L}_{\alpha,\omega}$ of the structure $L(\alpha)$, with elements x of $L(\alpha)$ assigned constant symbols $\underline{x}$.

(F2) The axioms of Σ_1 admissibility, viz., Extensionality, Foundation, Pairing, Union, Δ_0 Separation, and Δ_0 Bounding.

(F3) Let d be a new constant symbol.

$$d \text{ is an ordinal}$$

and for each ordinal $\beta < \alpha$,

$$d > \underline{\beta}.$$

(F4) Let $\underline{\mathcal{A}}$ be a new constant symbol.

$$\underline{\mathcal{A}} \text{ is a countable structure with underlying language } \mathcal{L}_{\alpha,\omega}, \text{ and}$$

for every formula $\vartheta \in \mathcal{T}_\alpha$

$$\underline{\mathcal{A}} \models \vartheta.$$

In particular, note that for each $\beta < \alpha$, (F1) contains the sentence

$$\forall x (x < \underline{\beta} \leftrightarrow \bigvee_{\gamma < \beta} (x = \underline{\gamma})),$$

which implies that any model of (F1) is an end extension of $L(\alpha)$.

The sentences (F1), (F2), and (F3) are all clearly Δ_1^α, and (F4) is also, because $\mathcal{T}_\alpha$ is Δ_1^α. Therefore, by Barwise compactness, (F) has a countable model $\mathcal{B}$. We will take $\mathcal{A}$ to be the structure $\mathcal{A}^\mathcal{B}$ denoted by the symbol $\underline{\mathcal{A}}$ in $\mathcal{B}$.

Sentences (F4) ensure that $\mathcal{A}^\mathcal{B}$ is a model of $\mathcal{T}_\alpha$. So the $\mathcal{L}_{\alpha,\omega}$-theory of $\mathcal{A}^\mathcal{B}$ is $\mathcal{T}_\alpha$, and hence *(2)* and *(6)* follow by Corollary 4.1. We can realize a non-principal type in some countable model of $\mathcal{T}_\alpha$ (even if not in one satisfying *(1)* or *(3)*) and so there is a non-atomic countable model of $\mathcal{T}_\alpha$. Once we have shown *(4)*, this will give us *(5)*.

By Lemma 1.4, the Scott rank of $\mathcal{A}^\mathcal{B}$ is at least α (one half of *(3)*). To obtain the rest of our claims, we modify the usual Henkin argument used to show Barwise compactness so as to satisfy

(i) $\alpha \notin \mathcal{B}$, and

(ii) $\mathcal{A}^\mathcal{B}$ realizes no non-principal types of $\mathcal{T}_\alpha$.

By (F2), any ordinal recursive in a real in $\mathcal{B}$ is already an element of $\mathcal{B}$, so (i) implies *(1)*. By (ii), the Scott rank of $\mathcal{A}^\mathcal{B}$ is not $\alpha + 1$, hence *(3)* and *(4)*.

Proof of (i). Consider a Δ_1^α Henkin construction which builds $\mathcal{B}$ in α many stages. For $\sigma < \alpha$, let $\mathcal{H}^\sigma$ be the theory with language $L_\mathcal{H}^\sigma$ determined in stage σ of the construction. We interleave the following two steps between each step of the construction:

Step (σ_a)*:* Suppose that after stage $\sigma < \alpha$ there is a constant $e \in L_\mathcal{H}^\sigma$ for which $\mathcal{H}^\sigma \cup \{e = \underline{\beta}\}$ is consistent for some $\beta < \alpha$. Then for the first such sentence $e = \underline{\beta}$ seen to be consistent after stage σ, enlarge (F) to include it.

Step (σ_b)*:* Suppose that after stage σ there is a constant e for which

$$\mathcal{H}^\sigma \vdash e \text{ is an ordinal}$$

and for each $\beta < \alpha$,

$$\mathcal{H}^\sigma \vdash \underline{\beta} < e.$$

Then let e' be a new constant and enlarge (F) to include the sentences

$$e' \text{ is an ordinal}$$

and for each $\beta < \alpha$,

$$\underline{\beta} < e' < e.$$

We now show that this axiom is consistent. Suppose not. This axiom and $\mathcal{H}^\sigma$ are both Σ_1^α, and so any contradiction which follows from them is a consequence of some α-finite subset. But then there is some ordinal $\beta_0 < \alpha$ for which

$$\bigwedge_{\beta < \beta_0} (\underline{\beta} < e' < e)$$

is contradictory. However, this is not contradictory by our hypothesis on e.

These steps guarantee (i), for if $\alpha \in \mathcal{B}$ then α is assigned a Henkin constant e introduced at some stage $\sigma < \alpha$. By some later stage $\sigma' < \alpha$ we have

$$\mathcal{H}^{\sigma'} \vdash e \text{ is an ordinal}$$

and for each $\beta < \alpha$,

$$(\underline{\beta} < e).$$

Then at step ${\sigma'}_b$, we add a constant which is realized in $\mathcal{B}$ by γ, say. By later steps τ_a for $\tau > \sigma$, we eventually produce an ordinal $\gamma \in \mathcal{B}$ with $\gamma < \alpha$ but also $\beta < \gamma$ for all $\beta < \alpha$, a contradiction.

Proof of (ii). Renumber the steps of the augmented Henkin construction from (i) so that steps are once again indexed by $\sigma < \alpha$. Suppose p is a non-principal n-type of $\mathcal{T}_\alpha$ realized by some n-tuple $b \in \mathcal{A}^{\mathcal{B}}$. Then at some stage $\sigma < \alpha$ there is a constant $\underline{b}$ for which

$$\mathcal{H}^\sigma \vdash \varphi(\underline{b})$$

for every $\varphi \in p$. But then p is Σ_1^α (as we may α-recursively enumerate the consequences of $\mathcal{H}^\sigma$), contradicting our hypothesis on $\mathcal{T}_\alpha$.

Thus for each non-principal n-type of $\mathcal{T}_\alpha$ and each n-tuple $b \in \mathcal{A}^{\mathcal{B}}$, there is a formula $\varphi(x) \in p$ for which $\neg\varphi(\underline{b})$ is consistent with $\mathcal{H}^\sigma$, for any choice of $\sigma < \alpha$. Let (p, b) denote a step in which we add one such $\neg\varphi(\underline{b})$ to (F). There are only countably many non-principal types in $\mathcal{T}_\alpha$, and $\mathcal{A}^{\mathcal{B}}$ is countable. Therefore we may interleave steps (p, b) with the first ω many steps of the original construction. (Note that we may additionally choose φ so that step (p, b) is consistent with all finitely many earlier such steps.) $\dashv$

5 Model (for $\omega_1 \leq \alpha < \omega_2$)

In this section, let $\omega_1 \leq \alpha < \omega_2$. Fix $\{T_n^\alpha : n < \omega\}$ as in Theorem 3.9. This time, by Theorem 2.9, the corresponding theory $\mathcal{T}_\alpha$ has some, but only $\aleph_1$ many, non-principal types, none of which are Σ_1^α. As before, by construction, $\mathcal{T}_\alpha$ is a complete and consistent Scott theory in the language $\mathcal{L}_{\alpha,\omega}$, as shown in Theorems 2.3 and 2.8. Thus we obtain

Corollary 5.1. *There is a complete and consistent Δ_1^α Scott theory $\mathcal{T}_\alpha$ in the language $\mathcal{L}_{\alpha,\omega}$ with some, but only $\aleph_1$ many, non-principal types, none of which are Σ_1^α.*

We use the following definitions and result from Sacks [17].

Definition 5.2. *Let $\mathcal{L}$ be a countable first-order language. Suppose A is a Σ_1 admissible set of cardinality $\aleph_1$ for which $\mathcal{L} \in A$. Let $\mathcal{L}_{A,\omega}$ be the restriction of $\mathcal{L}_{\infty,\omega}$ to formulas with standard codes in A. Suppose $T \subseteq A$. T is* amenable with respect to A *iff $(T \cap b) \in A$ for every $b \in A$.*

Definition 5.3. *Let $\mathcal{L}, A$, and T be as above.*

T is consistent *iff T is amenable with respect to A and no $\mathcal{L}_{\infty,\omega}$-deduction in A from T yields a contradiction.*

T is complete *iff for each sentence $\vartheta \in \mathcal{L}_{A,\omega}$, either $\vartheta \in T$ or $(\neg\vartheta) \in T$.*

A formula $\psi \in T$ is atomic *iff for every $\varphi \in \mathcal{L}_{A,\omega}$ either $(\psi \to \varphi) \in T$ or $(\psi \to (\neg\varphi)) \in T$.*

T is atomic *iff for each formula $\vartheta \in T$, there is a atomic formula $\psi \in T$ such that $(\psi \to \vartheta) \in T$.*

A model $\mathcal{M} \models T$ is atomic *iff every tuple of $\mathcal{M}^n$ satisfies some atomic formula of T (for $n < \omega$).*

Proposition 5.4. *Let $\mathcal{L}$ be a countable first-order language. Let A be a Σ_1 admissible set of cardinality $\aleph_1$. Suppose $T \subseteq \mathcal{L}_{A,\omega}$ and T is amenable with respect to A. Assume T is a consistent, complete, atomic theory with no countable atomic model. Then T has an atomic model of cardinality $\aleph_1$.*

Proof. See Sacks [17] Corollary 23 and Remark 2. ⊣

With somewhat stronger hypotheses we may modify this to realize a single additional type.

Proposition 5.5. *Let $\mathcal{L}$ be a countable first-order language. Let A be a Σ_1 admissible set of cardinality $\aleph_1$. Suppose $T \subseteq \mathcal{L}_{A,\omega}$ and T is amenable with respect to A. Assume T is a consistent, complete, atomic theory with no countable model, and let p be a type of T. Then T has an model of cardinality $\aleph_1$ realizing p.*

Proof. We may slightly modify the proof of Sacks [17] Corollary 23. Replace the set of atoms aT by $aT \cup \{p\}$. The theory is atomic, and so countable subsets of p are implied by some atom. The resulting model cannot be countable by hypothesis. ⊣

Using these two results, we can obtain a model of the appropriate Scott rank, though the resulting model might not preserve the admissibility of α.

Theorem 5.6. *There is a structure $\mathcal{A}$ of size $\aleph_1$ for which*

(1) the $\mathcal{L}_{\alpha,\omega}$-theory of $\mathcal{A}$ is Δ_1^α;

(2) the Scott rank of $\mathcal{A}$ is α;

(3) $\mathcal{A}$ is an atomic model of its $\mathcal{L}_{\alpha,\omega}$-theory;

(4) the $\mathcal{L}_{\alpha,\omega}$-theory of $\mathcal{A}$ is not $\aleph_1$-categorical; and

(5) no non-principal type of the $\mathcal{L}_{\alpha,\omega}$-theory of A is Σ_1^α.

Proof. Let $\mathcal{L}$ be the first-order language of $\mathcal{T}_\alpha$. It is countable, as it includes just a unary function f and the unary predicates $\{S_i : i < \omega\}$. The $|\alpha|$ many pseudo-predicates $\{U_{n,\beta}(\overline{x}_n) : n < \omega, \quad \beta < \alpha\}$ are actually defined recursively so that each is equivalent to a formula involving just f and the S_i's.

Let $A = \mathcal{L}_{\alpha,\omega}$. Note that A is a Σ_1 admissible set of cardinality $\aleph_1$ and $\mathcal{L} \in A$. Also note that $\mathcal{L}_{A,\omega} = \mathcal{L}_{\alpha,\omega}$.

Let $T = \mathcal{T}_\alpha$ from Corollary 5.1. T is amenable with respect to A as T is Δ_1^α. We also are given the consistency and completeness of T from Corollary 5.1 (the usual notions imply those of Definition 5.3).

If $\vartheta \in T$ then by the construction of the trees $\{T_n^\alpha : n < \omega\}$ there is a complete principal type extending ϑ; hence T is atomic. There are $\aleph_1$ many principal types of T, so no model is countable.

Let $\mathcal{A}$ be the atomic model of cardinality $\aleph_1$ given by Proposition 5.4. The $\mathcal{L}_{\alpha,\omega}$-theory of $\mathcal{A}$ is $\mathcal{T}_\alpha$ and so *(1)*, *(3)*, and *(5)* follow immediately. There are tuples of $\mathcal{A}$ with Scott rank unbounded below α, so the Scott rank of $\mathcal{A}$ is at least α. By *(3)* it is exactly α, and so we have *(2)*.

To see *(4)* we realize a non-principal type of $\mathcal{T}_\alpha$ (as given by Corollary 5.1) in a model $\mathcal{B}$ of $\mathcal{T}_\alpha$ of size $\aleph_1$ using Proposition 5.5. Since $\mathcal{A}$ is atomic and $\mathcal{B}$ is not, not all models of $\mathcal{T}_\alpha$ of size $\aleph_1$ are isomorphic. $\dashv$

Neither a Barwise compactness nor a Grilliot omitting-types argument seem to produce the desired extension of this result, viz., a model with the above properties which also preserves the admissibility of α. Perhaps a forcing argument may be useful in this connection.

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